Solutions to Sheet 3

Exercise 1

Let A be a PID. The arguments of $A = \mathbb{Z}$ from the lecture work verbatim to show that the prime ideals of $A[T]$ are

1. (0),
2. (f), $f \in A[T]$ irreducible,
3. (π, g) with $\pi \in A$ prime and $g \in A[T]$ a polynomial whose image in $(A/\pi)[T]$ is irreducible.

Show the following.

1. Assume that A has infinitely many prime ideals. Prove that the heights of the primes in (i), (ii) and (iii) are given by $0, 1, 2$ respectively. Show that each maximal ideal of $A[T]$ has height 2.
2. Let k be a field and set $A = k[\![u]\!]$. Show that $A[u^{-1}]$ is a field. Deduce that, in contrast to 1), the height 1 ideal $(uT - 1)$ is maximal.

Solution.

1. That (0) is of height zero is obvious. We showed on the last sheet that the only prime ideals contained in principal prime ideals (f) of UFDs are (0) and (f). As polynomial rings over UFDs are UFDs again, we are done with this case.

   The height of primes of the third form is at least 2. Indeed, we have inclusions $(0) \subset (\pi) \subset (\pi, g)$. We want to show that the height doesn't get larger than 2. The only thing that can go wrong is that there might be inclusions $(\pi, g) \subset (\pi', g')$.

   Assume we are given two prime ideals $\mathfrak{p} = (\pi, g) \subset (\pi', g') = \mathfrak{p}'$. By this inclusion we find $\mathfrak{p} \cap A = \mathfrak{p}' \cap A$, which shows $(\pi) = (\pi')$. But $A/(\pi)$ is a field, hence $A/(\pi)[T]$ is a PID and we find that the reductions of g and g' mod π are the same. This shows $(\pi, g) = (\pi, g')$, and we are done. (We have not used yet that there are infinitely many prime ideals).

   We also have to show that every maximal ideal is of the form (π, g). To this end, we have to show that every ideal of the form (f) is contained in some ideal (π, g). But if we write such f as $f = a_d T^d + \cdots + a_0$ and choose some prime $\pi \in A$ that does not divide a_d, we find that the reduction of f mod π is monic, at least up to multiplication with some unit. Hence we can choose some irreducible factor $g \in (A/\pi)[T]$ of f and lift it to a function $\tilde{g} \in A[T]$. We find that $(\pi, \tilde{g})$ is prime and contains (f), as desired.

2. We have $A[T]/(uT - 1) = A[u^{-1}]$. Note that A is a local ring and in particular a principal ideal domain (but with only a single prime). We have seen on a prior sheet that every element $x \in A \setminus (u)$ is invertible, hence we done.

Exercise 2

Let k be an algebraically closed field and let

$$\varphi : k[x,y] \to k[u,v], \quad x \mapsto u, \ y \mapsto uv.$$

1. Use exercise 1 to show that the maximal ideals of $k[x,y]$ are precisely the ideals
$$\mathfrak{m}_{\lambda,\mu} := (x-\lambda, y-\mu), \quad \lambda, \mu \in k.$$
2. Show that φ induces an isomorphism $k[x,y][x^{-1}] \to k[u,v][u^{-1}]$.
3. For each $(\lambda,\mu) \in k^2$ calculate $\operatorname{Spec}(\varphi)^{-1}(\mathfrak{m}_{\lambda,\mu})$.

Solution.

1. By exercise 1, the maximal ideals are precisely the ideals of the form (π, g) where $\pi \in k[x]$ is prime and $g \in k[x,y]$ is an element with reduction mod π is irreducible. As k is algberically closed, we find that $(\pi) = (x-\lambda)$ for some $\lambda \in k$. Now $k[x]/(x-\lambda) \cong k$, the isomorphism is given by $x \mapsto \lambda$. Hence $g(\lambda, y) \in k[y]$ needs to be irreducible, i.e., of the form $y - \mu$.
2. We can give an isomorphism $k[u,v][1/v] \to k[x,y][1/x]$ via $u \mapsto x$ and $v \mapsto y/x$. Checking that this is an isomorphism is straight-forward.
3. Note that $\operatorname{Spec}(\pi)^{-1}(\mathfrak{m}_{\lambda,\mu})$ is equal to the set of prime ideals $\mathfrak{p} \subset k[u,v]$ for which $\varphi(\mathfrak{m}_{\lambda,\mu}) \subset \mathfrak{p}$. By the homomorphism theorem, we find
$$\{\mathfrak{p} \subset k[u,v] \text{ prime} \mid \varphi(\mathfrak{m}_{\lambda,\mu}) \subset \mathfrak{p}\} \xrightarrow{1:1} \operatorname{Spec}(k[u,v]/\varphi(\mathfrak{m}_{\lambda,\mu})).$$
We find $\varphi(\mathfrak{m}_{\lambda,\mu}) = (u-\lambda, uv-\mu)$, hence
$$k[u,v]/\varphi(\mathfrak{m}_{\lambda,\mu}) \cong k[u,v]/(u-\lambda, uv-\mu) \cong k[v]/(v\lambda - \mu).$$
But this can be calculated explicitely:
$$k[v]/(v\lambda-\mu) \cong \begin{cases} k, & \text{if } \lambda \neq 0 \\ k[v], & \text{if } \lambda = 0 \text{ and } \mu = 0 \\ 0, & \text{if } \lambda = 0 \text{ and } \mu \neq 0 \end{cases}$$

Exercise 3

Let A be a ring of Krull dimension $n := \dim A$. Show that

$$n+1 \leq \dim A[T] \leq 2n+1.$$

Solution. Let
$$0 = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_m$$
be a maximal chain of prime ideals in $A[T]$. Write $\mathfrak{p}_i = \mathfrak{q}_i \cap A$. We obtain an ascending chain of prime ideals
$$0 = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_m$$

in A, which by the assumption on the dimension of A contains at most $n+1$ different prime ideals. We will show that $\mathfrak{p}_i = \mathfrak{p}_{i+1}$ implies $\mathfrak{p}_{i+1} \neq \mathfrak{p}_{i+2}$, which shows $m \leq 2n+1$.

More generally, we'll show that for every prime ideal $\mathfrak{p} \subset A$ the set of prime ideals $\mathfrak{q} \subset A[T]$ with $\mathfrak{q} \cap A = \mathfrak{p}$ (the primes above $\mathfrak{p}$) can have chains of length at most two. Let $\mathfrak{p} \subset A$ be such a prime ideal. For $S = A \setminus \mathfrak{p} \subset A \subset A[T]$ we make use of the bijection

$$\{\text{prime ideals in } S^{-1}A\} \xrightarrow{1:1} \{\text{prime ideals in } A \text{ not intersecting } S\},$$

which shows that there is no difference between primes above $\mathfrak{p}$ in A and in $A_{\mathfrak{p}} = S^{-1}A$. Hence we may assume that $\mathfrak{p}$ is maximal in A. Now $A/\mathfrak{p}$ is a field, hence $A/\mathfrak{p}[T]$ is a PID, hence of dimension 1. The inclusion-preserving bijection

$$\{\mathfrak{q} \subset A/\mathfrak{p}[T]\} \xrightarrow{1:1} \{\mathfrak{q} \subset A[T] \mid A[T]\mathfrak{p} \subset \mathfrak{q}\}$$

solves the exercise.

Exercise 4

Let A be a ring and $S, T \subset A$ multiplicative subsets with $S \subset T$.

1. Let $\iota_S : A \to S^{-1}A$ be the natural ring homomorphism. Show that $\iota_S^{-1}((S^{-1}A)^\times)$ is the saturation $\overline{S}$ of S.
2. Show that there exists a unique ring homomorphism $\iota : S^{-1}A \to T^{-1}A$ such that $\iota \circ \iota_S = \iota_T$.
3. Deduce that ι is an isomorphism if and only if $\overline{S} = \overline{T}$.

Solution.

First a reminder: The saturation of a subset $S \subset A$ is given by the set

$$\overline{S} = \{s \in A \mid \exists a \in A : as \in S\}.$$

1. First remember that ι_S is given by $a \mapsto \frac{a}{1}$. Now let's try to work out what the units in $S^{-1}A$ are. Remember that $S^{-1}A$ has underlying set

   $$(A \times S)/\sim_S, \quad \text{where} \quad (a, s) \sim_S (a', s') \text{ iff } as' = a's.$$

   In particular, we find that an element $(a, 1)$ $(= \frac{a}{1} = \iota_S(a))$ lies in the units of $S^{-1}A$ if and only if there are $a' \in A$, $s' \in S$ with $(aa', s') \sim_S (1, 1)$. This condition is equivalent to $(aa', 1) \sim_S (s', 1)$, which translates directly to what we had to show.

2. We define ι on representing objects using the inclusion $A \times S \to A \times T$. It is clear that this morphism behaves well under the equivalence relations $\sim_S$ and $\sim_T$ (here $\sim_T$ is defined the same way as for $\sim_S$), so we obtain a well-defined function

   $$S^{-1}A \cong (A \times S)/\sim_S \to (A \times T)/\sim_T \cong T^{-1}A.$$

   One readily checks that this indeed gives a map of rings (with addition and multiplication defined accordingly). One also readily checks that $\iota \circ \iota_S = \iota_T$.

3. We show that the saturation of S is maximal among the subsets $S \subset S' \subset A$ with $S'^{-1}A \cong S^{-1}A$ (where the induced morphism is given by ι). We first note that there is no difference between localizing at S and localizing at $\overline{S}$. Indeed, given some $s \in \overline{S}$, there is some $a \in A$ with $as \in S$. But now, given any $b \in A$, the element $\frac{b}{s} \in \overline{S}^{-1}A$ lies in the same equivalence class as $\frac{ba}{sa} \in S^{-1}A$. (Alternatively, this follows directly from what we showed in part 1: We have $\overline{S}^{-1}A = \overline{S}^{-1}(S^{-1}A) = S^{-1}A$, where we used in the last equality that $\overline{S} \subset (S^{-1}A)^{\times}$). Next, any subset $S \subset T \subset A$ that is not contained in $\overline{S}$ has non-isomorphic localization. Indeed, assume $t \in T \setminus \overline{S}$. Then the equivalence class of $\frac{1}{t} \in T^{-1}A$ does not lie in the image of ι by construction. Finally, note that whenever $S \subset T \subset \overline{S}$, we have $\overline{T} = \overline{S}$.

   This solves the exercise in an instant. For the one direction, if $\overline{S} = \overline{T}$, we find

$$S^{-1}A \cong \overline{S}^{-1}A = \overline{T}^{-1}A \cong T^{-1}A.$$

   For the other direction, if $S^{-1}A \cong T^{-1}A$, the result above directly implies $\overline{S} = \overline{T}$.

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