Solutions to Sheet 4

Exercise 1

Let A be a ring.

1. Assume that $f_n \in A[\![T]\!]$, $n \geq 0$, is a sequence of elements such that $f_n \in (T)^n$ for all $n \geq 0$. Show that there exists a unique element $f \in A[\![T]\!]$ such that $f - \sum_{k=0}^{n} f_k \in (T)^{n+1}$ for all $n \geq 0$.

2. Assume that A is noetherian. Show that $A[\![T]\!]$ is noetherian.

Solution.

1. We can just write down f. We need to find coefficients a_n such that $f = \sum_{n=0}^{\infty} a_n T^n$ satisfies $f - \sum f_k \in (T)^{n+1}$. Write $f_k = \sum_{j=0}^{k} a_{kj} T^j + (T)^k$. One quickly verifies that $a_n = \sum_{k=0}^{n} a_{kn}$ does the job.

2. Similar to the proof that the polynomial ring over a noetherian ring is noetherian, we let $I \subset A[\![T]\!]$ denote any ideal and denote by I' the ideal of A generated by the leading coefficients of functions in f, namely $I' := (a_d \mid f = a_d T^d + a_{d+1} T^{d+1} + \cdots \in A[\![T]\!])$. As A is noetherian, there is a finite number of elements $f_1, \ldots, f_n$ such that the leading (non-zero) coefficients of f_i gerate I'. Upon multiplying with powerst of T, we may assume that all f_i are of the form $f_i = a_{id} T^d + \ldots$ with $a_{id} \neq 0$ for some some suitable d.

   Now we claim that any $g \in I \cap T^d$ also lies in $(f_1, \ldots, f_n)$. Indeed, writing $g = b_d T^d + b_{d+1} T^{d+1} + \ldots$, we find that $b_d \in I'$, so we can eliminate the term $b_d T^d$ from g without leaving $I \cap T^d$. But now $g' = g - b_d T^d \in I \cap (T^{d+1})$. Upon repeatedly eliminating leading coefficients, we find $g \in (f_1, \ldots, f_n)$.

   To finish the argument, note that $A[\![T]\!]/(T^d) \cong A[T]/(T^d)$ is noetherian. Hence the image of I in this quotient is finitely generated, by $(g_1, \ldots, g_m)$, say. Choose lifts $(\tilde{g}_1, \ldots, \tilde{g}_m)$. Now, by construction, $I = (g_1, \ldots, g_m, f_1, \ldots, f_n)$.

Exercise 2

1. Let A be the ring of power series in $\mathbb{C}[\![z]\!]$ with a positive radius of convergence. Show that A is noetherian.

2. Show that the ring of holomorphic functions is not noetherian.

Solution.

1. One can quickly verify that all ideals of A are of the form (z^d). Indeed, every function that does not vanish at 0 does not have a root in some neighbourhood of 0 (by the identity theorem), hence admits a holomorphic inverse there. This shows that the units in A are given by $A \setminus (z)$. Now any non-unit is of the form $z^d u$ with u invertible and $d \geq 1$. The claim follows.

2. The hint commanded us to make use of the equation $\sin(2x) = 2\sin\cos(x)$. This shows that there is an infinite descending chain of ideals $(\sin(x)) \subset (\sin(x/2)) \subset (\sin(x/4)) \subset \dots$. It is clear that this chain does not get stationary, by looking at the real roots of those functions.

Exercise 3

Let $n \geq 1$. For an $n \times n$ matrix M over some ring A denote by $\chi_M(T) = \det(T \cdot \mathrm{Id} - M)$ its characteristic polynomial.

1. Let $A = \mathbb{Z}[a_{ij} \mid 1 \leq i, j \leq n]$ and $M \coloneqq (a_{ij})_{ij} \in \mathrm{Mat}_n(A)$. Show that $\chi_M(M) = 0$.
2. Deduce a general form of the theorem of Cayley-Hamilton: Let A be a ring and let M be any $n \times n$ matrix over A. Then $\chi_M(M) = 0$.

Solution.

1. Since A is integral, we can pass to the field of fractions of A. Now the regular cayley hamilton applies. (Note that the calculation of the determinant does not depend on whether we are in the field of fractions or not).
2. There is a surjective map $\pi : \mathbb{Z}[a \in A] \to A$ given by $a \mapsto a$. By part 1 we find that $\chi_M(M) = 0$ in $\mathbb{Z}[a \in A]$. Now $0 = \pi(\chi_M(M)) = \chi_M(M)$. Done?

Exercise 4

Let A be a principal ideal theorem.

1. Let $a \in A \setminus \{0\}$ and let $\pi \in A$ be prime. Set $B \coloneqq A/(a)$. For any $n \geq 0$ show that

$$\dim_{A/(\pi)} \pi^n B/\pi^{n+1}B = \begin{cases} 0, & \text{if } \nu_\pi(a) \leq n \\ 1, & \text{if } \nu_\pi(a) \geq n+1. \end{cases}$$

2. Assume that $M = A^r \oplus A/(a_1) \oplus \cdots \oplus A/(a_k)$, $N = A^s \oplus A/(b_1) \oplus \cdots \oplus A/(b_l)$ with $a_i, b_i \in A$ non zero and $a_1 \mid a_2 \mid \cdots \mid a_k$, $b_1 \mid \cdots \mid b_l$. Show that if $M \cong N$, then $r = s$, $k = l$ and $(a_i) = (b_i)$ for all i.

Solution.

1. First, note that there are isomorphism (of $A/(\pi)$ vector spaces)

$$\pi^n B/\pi^{n+1}B \cong \pi^n \left(A/(a)\right)/\pi^{n+1}\left(A/(a)\right) \cong \pi^n A/((\pi^{n+1}, a) \cap (\pi^n)).$$

We have seen before that for principal ideal domains, we have $\dim_{A/(\pi)}(\pi^n A)/(\pi^{n+1}A) = 1$. The right hand side of the equation above is the same as $\pi^n A/\pi^{n+1}A$ if $a \in (\pi^{n+1})$, otherwise we quotient out by some non-zero subspace. Hence we see that the quotient above vanishes if and only if $a \notin \pi^{n+1}$, that is, if and only if $\nu_\pi(a) \geq n+1$.

2. The exercise is confusing, because depending on how one thinks about modules over rings (especially if one thinks of them as generalized vector spaces) it might seem tautological. The main problem is that plain isomorphisms don't necessarily respect direct sums. Over non-commutative rings, there even are examples of modules S for which $S \oplus S \cong S$! Over PIDs however, everything seems to be well-behaved. We solve the exercise in two steps.
Step 1: $r = s$. Let $K = \mathrm{Frac}(A)$ denote the field of fractions of A. Then upon tensoring with K, the torsion part of M and N vanishes. Now by $M \cong N$, we find $K^r \cong K \otimes M \cong K \otimes N \cong K^s$. As every basis of a finite dimensional vector space has the same number of elements, this shows $r = s$. Appearently you have also already seen this in the lecture.
Step 2: The torsion part. We fix some prime element $\pi \in A$ and use part 1 of the exercise. The isomorphism $M \cong N$ yields an isomorphism $\pi^n M/\pi^{n+1} M \cong \pi^n N/\pi^{n+1} N$. Choosing $n = \nu_\pi(a_k)$, this shows that the π-adic valuations of a_k and b_l agree. Moreover, the number

$$d_n = \dim_{A/(\pi)} \pi^n M/\pi^{n+1} M = \dim_{A/(\pi)} \pi^n N/\pi^{n+1} N$$

is the number of elements among the a_i such that $\nu_\pi(a_i) = n$. Iterating over all prime numbers π, this shows that

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