Solutions to Sheet 5

Exercise 1

Let A be a ring and let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n \subset A$ be ideals such that $\bigcap_{i=1}^n \mathfrak{a}_i = \{0\}$. Assume that each ring $A/\mathfrak{a}_i$ is noetherian. Show that A is noetherian.

Solution. Let $\pi_i : A \to A/\mathfrak{a}_i$ denote the projections. We have the map

$$\pi = (\pi_1, \ldots, \pi_n) : A \to A/\mathfrak{a}_1 \times \cdots \times A/\mathfrak{a}_n.$$

As the $\mathfrak{a}_i$ have intersection $\{0\}$, π is injective. Hence A is isomorphic to the subring $\mathrm{Im}(\pi) \subset A/\mathfrak{a}_1 \times \cdots \times A/\mathfrak{a}_n$. This shows that A is isomorphic to the subring of a noetherian ring, thereby noetherian.

Exercise 2

Consider the matrix

$$S := \begin{pmatrix} -36 & 14 & -24 \\ 18 & 6 & 12 \end{pmatrix}.$$

Determine its elementary divisors and the kernel/cokernel of the map $\mathbb{Z}^3 \xrightarrow{S} \mathbb{Z}^2$ (up to isomorphy).

Solution. We want to find simpler representatives of the residue class of S in the double quotient $\mathrm{GL}_2(A) \backslash \mathrm{Mat}_{2\times 3}(A) / \mathrm{GL}_3(A)$. We add twice the lower row to the upper row (which is the same as multiplying by $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ from the left), which gives

$$S \sim \begin{pmatrix} 0 & 26 & 0 \\ 18 & 6 & 12 \end{pmatrix}.$$

Further transformations yield

$$\begin{pmatrix} 0 & 26 & 0 \\ 18 & 6 & 12 \end{pmatrix} \to \begin{pmatrix} 0 & 26 & 0 \\ 6 & 6 & 12 \end{pmatrix} \to \begin{pmatrix} 0 & 26 & 0 \\ 6 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 6 & 0 & 0 \\ 0 & 26 & 0 \end{pmatrix}.$$

This allows us to calculate kernel and cokernel of S. We find

$$\mathrm{Ker}(S) \cong \mathbb{Z}, \quad \mathrm{Coker}(S) \cong \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/26\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/78\mathbb{Z}.$$

This shows that the elementary divisors are given by 2 and 78.

Exercise 3

Let A be a ring, let $\mathfrak{a} \subset A$ be an ideal and let M, N_i, $i \in I$, be A-modules for some set I.

1. Show that there exists a unique isomorphism

$$\Phi : \bigoplus_{i \in I} (N_i \otimes_A M) \to \left(\bigoplus_{i \in I} N_i \right) \otimes_A M$$

such that $\Phi((\ldots, 0, n_i \otimes m, 0 \ldots)) = (\ldots, 0, n_i, 0, \ldots) \otimes m$ for all $n_i \in N_i$, $i \in I$, $m \in M$.

2. Show that there exists a unique isomorphism

$$\Psi : A/\mathfrak{a} \otimes_A M \to M/\mathfrak{a}M$$

such that $\Psi((a+\mathfrak{a}) \otimes m) \mapsto am + \mathfrak{a}M$ for all $a \in A$, $m \in M$.

Solution. This exercise looks like you'd have to do lots of calculations, but there is the following rule:

NEVER DO ANYTHING EXPLICITLY WHEN WORKING WITH TENSOR PRODUCTS.

1. We could try to solve this by somehow checking that the map is well-defined, working everything out element-wise, and in the end showing that the isomorphism we obtain is somehow unique. But this is messy, and probably confusing to anyone who wants to follow the argument. It is much cleaner to work with universal properties. Note that $\bigoplus_{i\in I}(N_i \otimes_A M)$ satisfies the following universal property:

   For any A-module P and any tuple of bilinear maps $(\varphi_i : N_i \times M \to P)_i$,
   there is a unique linear map $\Phi : \bigoplus_{i\in I}(N_i \otimes M) \to P$ such that $\Phi(n_i \otimes m) = \varphi_i(n_i, m)$.

   That $\bigoplus_{i\in I}(N_i \otimes_A M)$ satisfies this universal property is easy to see. The UP of the tensor product gives linear maps $N_i \otimes M \to P$ associated to φ_i, and we obtain φ by the UP of the direct sum. But note that $(\bigoplus_{i\in I} N_i) \otimes M$ satisfies the same UP. Indeed, one easily checks that a tuple of bilinear maps $(\varphi_i : N_i \times M \to P)_{i\in I}$ is the same data as a single bilinear map $(\varphi : (\bigoplus_{i\in I} N_i) \times M \to P)$. This automatically gives a unique isomorphism

   $$(\bigoplus_{i\in I} N_i) \otimes M \cong \bigoplus_{i\in I}(N_i \otimes M),$$

   which is of the desired form by construction.

2. I lied to you, this time we do things explicitely. The mapping

   $$A/\mathfrak{a} \times M \to M/\mathfrak{a}M, \quad (a+\mathfrak{a}, m) \mapsto am + \mathfrak{a}M.$$

   is well-defined and bilinear, which is easy to check. This gives the desired map $\Psi : A/\mathfrak{a} \otimes_A M \to M/\mathfrak{a}M$. It is surjective as $\Psi(1 \otimes m) = m + \mathfrak{a}M$, and injective because if $\Psi((a+\mathfrak{a}) \otimes m) = 0 + \mathfrak{a}M$, we have $am \in \mathfrak{a}M$. Hence $am = a'm'$ for some $a' \in \mathfrak{a}, m' \in M$. In particular,

   $$a \otimes m = 1 \otimes (am) = 1 \otimes (a'm') = a' \otimes m' = 0 \in A/\mathfrak{a} \otimes_A M.$$

   This shows injectivity of Ψ, and we are done.

   HAHAHA FOOLS! The proof above doesn't work! Namely, to show injectivity, it does not suffice to check that there are no nontrivial elements of the form $a \otimes m$ that get sent to zero. There might still be linear combinations of such elements which are getting sent to zero. But showing that $\sum a_i \otimes m_i \mapsto 0 \implies \sum a_i \otimes m_i = 0$ is really hard, there is no way to get a handle on the sum.

   So we try UPs again. We show that for any bilinear map $(-,-) : A/\mathfrak{a} \times M \to P$ there is a unique linear map $\varphi : M/\mathfrak{a}M \to P$ with $\varphi(am) = (a, m)$. This can be checked directly.

Exercise 4

Let A be a ring and let M, N be A-modules. A bilinear map $(-,-) : M \times M \to N$ is called symmetric if $(m_1, m_2) = (m_2, m_1)$ for all $m_1, m_2 \in M$. It is called alternating if $(m, m) = 0$ for all $m \in M$.

1. Show that there exists an A-module $\mathrm{Sym}^2_A(M)$ and a symmetric bilinear map $\iota : M \times M \to \mathrm{Sym}^2_A(M)$ with the following universal property: For every A-module N and for every symmetric bilinear map $(-,-) : M \times M \to N$ there exists a unique A-linear map $\Phi : \mathrm{Sym}^2_A(M) \to N$ usch that for all $m_1, m_2 \in M$

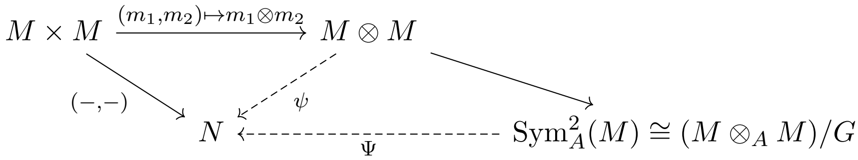
$$(m_1, m_2) = \Psi(\iota(m_1, m_2)).$$

   Construct similarly an A-module $\Lambda^2_A(M)$ with a universal alternating bilinear map $\gamma : M \times M \to \Lambda^2_A(M)$.

2. Show that $\mathrm{Sym}^2_A(A^n)$ and $\Lambda^2_A(A^n)$ are free A-modules of ranks $\frac{n(n+1)}{2}$ and $\frac{n(n-1)}{2}$.

Solution.

1. Okay, the Sym-construction should be somehow similar to the construciton of $\otimes$, and ideally all proofs of properties simply follow from the universal property of the tensor product. In the construction of the tensor product, (m_1, m_2) corresponds to the image of $\varphi(m_1 \otimes m_2)$ for some suitable morphism φ. Imposing that $(m_1, m_2) = (m_2, m_1)$ corresponds to the statement that in Sym^2_A, any morphism should send $(m_1 \otimes m_2 - m_2 \otimes m_1)$ to zero. Building on this, we define $\mathrm{Sym}^2_A(M)$ as $(M \otimes_A M)/G$, where G is the A-module generated by elements of the form $(m_1 \otimes m_2 - m_2 \otimes m_1)$. We check that this works. With the notation of the exercise, we first obtain a morphism $\psi : M \otimes_A M \to N$ by the UP of the tensor product.



   By construction, we have $G \subset \mathrm{Ker}\,\psi$, so by the universal property of kernels, ψ extends uniquely to a morphism $\Psi : \mathrm{Sym}^2_A(M) \cong (M \otimes_A M)/G \to N$.

   We define $\Lambda^2_A(M)$ similarly, this time we define G as submodule of $M \otimes_A M$ generated by elements of the form $(m \otimes m)$.

2. We'll again first focus on Sym^2_A. First of all, note that the set of bilinear maps $(-,-) : A^n \times A^n \to N$ with values in an A-module N is the same as the set of matrices $(a_{ij})_{i,j=1,\dots,n}$ with $a_{ij} \in N$. The argument essentially comes from linear algebra; we simply associate to $(-,-)$ the matrix $((e_i, e_j))_{ij}$. Now, note that the subset of symmetric bilinear forms corresponds to those matrices with $a_{ij} = a_{ji}$. The set of these matrices has a natural structure of a free A-module of rank $\frac{n(n+1)}{2}$. We need to show that this number is equal to the rank of Sym^2_A. But for any A-module N, we have established the isomorphisms

$$\begin{aligned} N^{\frac{n(n+1)}{2}} &\cong \{M = (a_{ij})_{ij} \mid a_{ij} \in N \text{ and } a_{ij} = a_{ji}\} \\ &\cong \mathrm{SymBiHom}(A^n, A^n; N) \cong \mathrm{Hom}_A(\mathrm{Sym}^2_A(A^2), N). \end{aligned}$$

Here, $\mathrm{SymBiHom}(A^n, A^n; N)$ denotes the space of symmetric bilinear maps $A^n \times A^n \to N$. The functor sending N to $N^{\frac{n(n+1)}{2}}$ is represented by $A^{\frac{n(n+1)}{2}}$. Hence, utilizing the Yoneda-lemma, we find that $A^{\frac{n(n+1)}{2}} \cong \mathrm{Sym}^2_A(A^n)$.

For $\Lambda^2_A(A^n)$, we do exactly the same. The only thing that changes is the set of matrices we look at, as this time we have isomorphisms

$$\{M = (a_{ij})_{ij} \mid a_{ij} \in N \text{ and } a_{ij} = -a_{ji} \text{ and } a_{ii} = 0\} \cong \mathrm{AltBiHom}_A(A^n, A^n, N).$$

The space of matrices is quickly seen to be isomorphic to $N^{\frac{n(n-1)}{2}}$.

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