Solutions to Sheet 7

Exercise 1

Let $A \to B$ be a homomorphism of rings, let M be an A-module and let N be a B-module.

1. Show that the map

$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_B(B \otimes_A M, N), \quad \varphi \mapsto (b \otimes m \mapsto b\varphi(m))$$

   is a well-defined isomorphism.

2. Show that the map

$$M \otimes_A N \to (M \otimes_A B) \otimes_B N, \quad m \otimes n \mapsto (m \otimes 1) \otimes n$$

   is a well-defined isomorphism.

3. Deduce that $S^{-1}M_1 \otimes_A S^{-1}M_2 \cong S^{-1}M_1 \otimes_{S^{-1}A} S^{-1}M_2$ for two A-modules M_1, M_2 and a multiplicative subset $S \subset A$.

Solution.

1. A function $\varphi \in \mathrm{Hom}_B(B \otimes_A M, N)$ is uniquely determined by its values on elementary tensors. We have $\varphi(b \otimes m) = b\varphi(1 \otimes m)$, so in reality φ is uniquely determined by its values on $1 \otimes m$. But any such morphism gives rise to a A-linear map via $m \mapsto 1 \otimes m \mapsto \varphi(1 \otimes m)$, and conversely any $\psi \in \mathrm{Hom}_A(M, N)$ yields a unique morphism via $b \otimes m \mapsto b\psi(m) \in \mathrm{Hom}_B(B \otimes_A M, N)$. These constructions are quickly checked to be mutually inverse.

   Remark. This is a special case of the so called *Hom-Tensor adjunction.* It states that there is a natural isomorphism

$$\mathrm{Hom}_B(M \otimes_A L, N) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_B(L, N)).$$

   In more fany terms, this says that the functors $\mathrm{Hom}_B(L, -) : \mathsf{Mod}_B \to \mathsf{Mod}_A$ and $- \otimes_A L : \mathsf{Mod}_A \to \mathsf{Mod}_B$ is an adjoint pair, for any B-module L.

2. Again, universal properties. Of course, we'll want to show that this is an isomorphism of B-modules. We do this by using the universal property. What is a B-linear morphism $\varphi : (M \otimes_A B) \otimes_B N \to P$? The same as a B-bilinear map $\Phi : (M \otimes_A B) \times N \to P$. But as $\Phi(m \otimes b, n) = b\Phi(m \otimes 1, n)$, any such bilinear map is uniquely determined by its values on elements of the form $(m \otimes 1, n)$, hence it really is the same as a A-bilinear map $M \times N \to P$, given by $(m, n) \mapsto (m \otimes 1, n) \mapsto \Phi(m \otimes 1, n)$. This construction is quickly verified to be an isomorphism. But now $(M \otimes_A B) \otimes_B N$ satisfies the universal property of $M \otimes_A N$.

3. We apply the above with $S^{-1}M_1 = M$ and $S^{-1}M_2 = N$ and $B = S^{-1}A$. Note that $S^{-1}M_1 \otimes_A S^{-1}A \cong S^{-1}(S^{-1}M_1) \cong S^{-1}M_1$, which gives (following the above)

$$S^{-1}M_1 \otimes_A S^{-1}M_2 \cong (M \otimes_A S^{-1}A) \otimes_{S^{-1}A} S^{-1}M_2 \cong S^{-1}M_1 \otimes_{S^{-1}A} S^{-1}M_2.$$

Exercise 2

Let A be a ring. We define the *support* of an A-module M as $\operatorname{Supp}(M) := \{\mathfrak{p} \in \operatorname{Spec}(A) \mid M_\mathfrak{p} \neq 0\}$.

1. Assume M is finitely generated. Show that $\operatorname{Supp}(M) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid M \otimes_A k(\mathfrak{p}) \neq 0\}$, where $k(\mathfrak{p}) = \operatorname{Quot}(A/\mathfrak{p})$.
2. Assume M, N are finitely generated A-modules. Show $\operatorname{Supp}(M \otimes_A N) = \operatorname{Supp}(M) \cap \operatorname{Supp}(N)$.

Solution.

1. We will show that $M_\mathfrak{p} \neq 0$ if and only if $M \otimes_A k(\mathfrak{p}) \neq 0$. The map $A \to k(\mathfrak{p})$ factors through the map $A_\mathfrak{p} \to k(\mathfrak{p})$, and we find $M \otimes_A k(\mathfrak{p}) = M_\mathfrak{p} \otimes_{A_\mathfrak{p}} k(\mathfrak{p})$, this directly gives the implication $M \otimes_A k(\mathfrak{p}) \neq 0 \implies M_\mathfrak{p} \neq 0$.

   For the other direction, we use Nakayama's Lemma. It (or at least one version of it) states that if $N \neq 0$ is a finitely generated module over a local ring B with maximal ideal I, we have $IN \neq N$. In our situation, if we assume $M_\mathfrak{p} \neq 0$, Nakayama says

   $$M_\mathfrak{p} \otimes_{A_\mathfrak{p}} k(\mathfrak{p}) \cong M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \neq 0.$$

   Done.

2. We'll show that $(M \otimes_A N) \otimes k(\mathfrak{p}) \neq 0$ if and only if $M \otimes_A k(\mathfrak{p}) \neq 0$ and $N \otimes_A k(\mathfrak{p}) \neq 0$. Exercise 1.2 gives the isomorphism

   $$(M \otimes_A k(\mathfrak{p})) \otimes_{k(\mathfrak{p})} (N \otimes_A k(\mathfrak{p})) \cong M \otimes_A (N \otimes_A k(\mathfrak{p})) \cong (M \otimes_A N) \otimes_A k(\mathfrak{p}).$$

   From here we can directly check the desired equivalence.

Exercise 3

Let A be a ring, let $S \subset A$ be a multiplicative subset and let M, N be A-modules.

1. Assume that M is finitely presented A-module. Show that the map

   $$S^{-1} \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N), \quad \varphi/s \mapsto (m/t \mapsto \varphi(m)/st)$$

   is a well-defined isomorphism.
2. Construct a counterexample to the above if M is only assumed to be finitely generated.

Solution.

1. First, note that we always (without hypothesis on M) obtain such a map. This follows (for example) from exercise 1.1 with $B = S^{-1}A$. We obtain the isomorphism

   $$\operatorname{Hom}_A(M, S^{-1}N) \xrightarrow{\sim} \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N).$$

Also, the natural map $N \to S^{-1}N$ yields a map

$$\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(M, S^{-1}N).$$

Finally, as multiplication with any $s \in S$ gives an isomorphism on the right hand side, we obtain a morphism

$$S^{-1}\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(M, S^{-1}N) \xrightarrow{\sim} \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N).$$

by the universal property of localization on modules. One readily checks that this morphism is the one provided by the exercise.

Now we have to show that this is an isomorphism if M is finitely presented. As usual, we write M as part of a short exact sequence

$$0 \to A^m \to A^n \to M \to 0.$$

Now we use that $\operatorname{Hom}_A(-, N)$ is right-exact. Hence applying $\operatorname{Hom}_A(-, N)$ yields an exact sequence

$$0 \to 0 \to \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(A^n, N) \cong N^n \to \operatorname{Hom}_A(A^m, N) \cong A^m.$$

Localizing at S is exact, so we obtain

$$0 \to 0 \to S^{-1}\operatorname{Hom}_A(M, N) \to (S^{-1}N)^n \to (S^{-1}N)^m.$$

Similarly, we can localize at S first and then apply $\operatorname{Hom}_{S^{-1}A}(S^{-1}(-), S^{-1}N)$, which yields the exact sequence

$$0 \to 0 \to \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \to (S^{-1}N)^n \to (S^{-1}N)^m.$$

Now we can use the 5-lemma again!

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & S^{-1}\operatorname{Hom}_A(M, N) & \longrightarrow & (S^{-1}A)^n & \longrightarrow & (S^{-1}A)^m \\
\| & & \| & & \downarrow{\scriptstyle \therefore \text{iso}} & & \| & & \| \\
0 & \longrightarrow & 0 & \longrightarrow & \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) & \longrightarrow & (S^{-1}A)^n & \longrightarrow & (S^{-1}A)^m
\end{array}$$

2. A standard example seems to be the following. Let $A = k[x, y_1, y_2, \ldots]$ be the polynomial ring in variables indexed by $\mathbb{N}$. Let $M = A/(y_1, y_2, \ldots)$, $N = A/(xy_1, x^2y_2, \ldots)$ and $S = \{1, x, x^2, \ldots\}$. Now let's compare both sides of the morphism. Note that M is generated by 1, so that any A-linear morphism $\varphi : M \to N$ is uniquely determined by the value of $\varphi(1) \in N$. Now we have $0 = y_1\varphi(1) = y_2\varphi(1) = \ldots$, which shows that any lift $\widetilde{\varphi(1)} \in R$ is infintely divisible by x, hence $\varphi(1) = 0$. On the left hand side, we find that $S^{-1}M \cong S^{-1}N \cong k[x^{\pm 1}]$, so there are many $S^{-1}A$-linear morphisms $S^{-1}M \to S^{-1}N$.

Exercise 4

Let A be a principal domain and let $f \in A \setminus \{0\}$ be a non-unit. Show that the $A[T]$-module $(f, T) \subset A[T]$ is not flat.

Solution. Consider the map given by multiplication with f, which we will denote as $\varphi : A \to A$. It is injective. Note that $A \cong A[T]/(T)$. We want to show that $(f, T) \otimes_{A[T]} \varphi$ is not injective, showing that (f, T) is not flat. We have an isomorphism (of $A[T]$-modules)

$$(f, T) \otimes_{A[T]} A \cong (f, T)/T(f, T),$$

and $(f, T) \otimes \varphi$ corresponds to the endomorphism given by multiplication with f under this identification. Now, $T \neq 0$ in $(f, T)/T(f, T)$, but $fT = \varphi(T) = 0$.

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