Solutions to Sheet 14

Exercise 1

Let $m \in \mathbb{N}$ and let χ be a Dirichlet character modulo m.

1. Show that $\log|1-w| = -\operatorname{Re}\sum_{k=1}^{\infty}\frac{w^k}{k}$ for all $w \in \mathbb{C}$ with $|w| < 1$.
2. Show that $3 + 4\cos\theta + \cos 2\theta \geq 0$ for all $\theta \in \mathbb{R}$.
3. Show that $|1-w|^3\,|1-wu|^4\,|1-wu^2| \leq 1$ for all $w \in [0,1)$ and all $u \in \mathbb{C}$ with $|u| = 1$.
4. Show that $|\zeta(\sigma)^3|\,|L(\sigma+it;\chi)|^4\,|L(\sigma+2it;\chi^2)| \geq 1$ for all $\sigma \in (1,\infty)$ and $t \in \mathbb{R}$.
5. Deduce that $L(1+it;\chi) \neq 0$ for all $t \in \mathbb{R}\setminus\{0\}$.

Solution. This exercise reviews standard methods to prove zero-free regions of L-functions, and most introductors texts to analytic number theory should cover the content of this exercise (see for example [Jörg Brüdern, *Einführung in die analytische Zahlentheorie*, p. 101f]). Hence I will only sketch the solution.

1. The power series is that of the standard branch of the logarithm. The claim now follows by its properties.

2. This is a trick found by Hadamard and de la Vallée Poussin. One quickly checks that

$$3 + 4\cos\alpha + \cos 2(\alpha) = 2(1+\cos\alpha)^2 \geq 0.$$

3. Okay clearly the previous two exercises want us to take logaritm. We write $u = \mathrm{e}^{\alpha \mathrm{i}}$ and find

$$\begin{aligned}\log(|1-w|^3\,|1-wu|^4\,\left|1-wu^2\right|) &= 3\log(|1-w|) + 4\log|1-wu| + \log(\left|1-wu^2\right|)\\ &= -\sum_k \frac{w^n}{n}\left(3+4\cos(n\alpha)+\cos(2n\theta)\right) \leq 0.\end{aligned}$$

In the last inequality we used 2.

4. Developing this in a euler product, the terms that occur are exactly of the form from exercise 3, but (multiplicatively) inverted. The claim follows.

5. Suppose that $L(1+\mathrm{i}t;\chi) = 0$ for some $t \neq 0$. Then also

$$\lim_{\sigma\to 1^+}\left|\zeta(\sigma)^3\right|\,|L(\sigma+it;\chi)|^4\,\left|L(\sigma+2it;\chi^2)\right| = 0,$$

as all the functions are analytic the degree 3-pole of $\zeta(\sigma)^3$ at $\sigma = 1$ get's eaten by $L(\sigma+\mathrm{i}t;\chi)^4$, which is a degree-4-zero at this point. This contradicts part 4. (Here we used that all the functions have (meromorphic) continuations to the whole plane.)

Exercise 2

Let χ be a non-trivial Dirichlet character modulo 8. Show that

$$L(1;\chi) > 1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} = \frac{64}{105}.$$

Solution. The idea is that every Dirichlet character mod 8 is real as every element in $(\mathbb{Z}/2\mathbb{Z})$ has order at most 2. Furthermore, we have $\chi(3)\chi(5) = \chi(7)$. Hence the smallest possible value that the series

$$L(1;\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} = \sum_{k\in 8\mathbb{N}} \left(\frac{\chi(k+1)}{k+1} + \frac{\chi(k+3)}{k+3} + \frac{\chi(k+5)}{k+5} + \frac{\chi(k+7)}{k+7} \right)$$

can ever take is if $\chi(3) = \chi(5) = (-1)$ and $\chi(7) = 1$. In this case the above becomes

$$L(1;\chi) = \sum_{k\in 8\mathbb{N}} \left(\frac{1}{k+1} - \frac{1}{k+3} - \frac{1}{k+5} + \frac{1}{k+7} \right)$$

One quickly checks that for each $k \in \mathbb{N}$ (as $x \mapsto \frac{1}{x}$ is a convex function),

$$\frac{1}{k+1} - \frac{1}{k+3} - \frac{1}{k+5} + \frac{1}{k+7} > 0,$$

and the claim follows after truncating the series above at $k = 1$.

Exercise 3

Let $m \in \mathbb{N}$, set $\zeta_m = \mathrm{e}^{2\pi\mathrm{i}/m}$ and let $K \subset \mathbb{Q}(\zeta_m)$ be a number field of degree d. Set $G = \mathrm{Gal}(K/\mathbb{Q})$ and identify $\widehat{G}$ with a subgroup of the group of Dirichlet characters modulo m as in the lecture.

1. Show that for each $\chi \in \widehat{G}$, there exists a unique $f = f_\chi \in \mathbb{N}$ (called the *conductor* of χ) such that $f \mid m$ and χ is the composition of the canonical homomorphism $(\mathbb{Z}/m\mathbb{Z})^\times \to (\mathbb{Z}/f\mathbb{Z})^\times$ with a primitive Dirichlet character χ^{prim} modulo f.
2. Let p be a prime number such that $m = m'p^e$ for some $m', e \in \mathbb{N}$ such that $p \nmid m'$. Set $\zeta_{m'} = \mathrm{e}^{2\pi\mathrm{i}/m'}$ and $L = K \cap \mathbb{Q}(\zeta_{m'})$. Show that
$$\prod_{\mathfrak{p}\mid p\mathcal{O}_K} (1 - \mathrm{N}(\mathfrak{p})^{-s}) = \prod_{\mathfrak{q}\mid p\mathcal{O}_L} (1 - \mathrm{N}(\mathfrak{q})^{-s}) = \prod_{\chi\in\widehat{G}} (1 - \chi^{\mathrm{prim}}(p)p^{-s})$$
3. Show that $\zeta_K(s) = \prod_{\chi\in\widehat{G}} L(s;\chi^{\mathrm{prim}})$ for all $s \in \mathbb{C}\setminus\{1\}$ with $\mathrm{Re}\, s > 1 - 1/d$.

Solution.

1. We let $f > 1$ be the minimal integer with the desired property.
2. We are in the following situation.

$$\begin{array}{ccc}
K & \longrightarrow & \mathbb{Q}(\zeta_m) = \mathbb{Q}(\zeta_{p^e})\cdot\mathbb{Q}(\zeta_{m'}) \\
\Big| & & \uparrow \\
L & \longrightarrow & \mathbb{Q}(\zeta_{m'}) \\
\Big| & & \uparrow \\
\mathbb{Q} & \longrightarrow & \mathbb{Q}
\end{array}$$

We know that the extension $\mathbb{Q} \hookrightarrow \mathbb{Q}(\zeta_m')$ is unramified above p as p does not divide the discriminant of $\mathbb{Q}(\zeta_{m'})$. Hence p is also unramified in L (by multiplicativity of ramification indeces along extensions of fields), and we find that

$$p\mathcal{O}_K = \mathfrak{q}_1 \dots \mathfrak{q}_r$$

for r pairwise distinct primes $\mathfrak{q}_i \subset \mathcal{O}_L$. Let f_i be the residue degree of $\mathfrak{q}_i$ over p, then on the side of the Euler factors we find

$$\prod_{\mathfrak{q} | p\mathcal{O}_L} (1 - \mathrm{N}(\mathfrak{q})^{-s}) = \prod_{i=1}^{r} (1 - p^{-f_i s}).$$

As $\mathbb{Q}(\zeta_m) \cong \mathbb{Q}(\zeta_{m'})[T]/(T^{p^e} - 1)$ we find that above p (or rather, above every prime dividing $p\mathcal{O}_{\mathbb{Q}(\zeta_{m'})}$) the extension $\mathbb{Q}(\zeta_{m'}) \hookrightarrow \mathbb{Q}(\zeta_m)$ is totally ramified, hence the extension $L \hookrightarrow K$ is totally ramified as well (by multiplicativity of residue degrees along extensions). Thereby we can write $\mathfrak{q}_i \mathcal{O}_K = \mathfrak{p}_i^{e_i}$ for prime ideals $\mathfrak{p}_i \subset \mathcal{O}_K$. But the Euler factors *forget* about the numbers e_i. We obtain the first equality, as now

$$\prod_{\mathfrak{p} | p\mathcal{O}_K} (1 - \mathrm{N}(\mathfrak{p})^{-s}) = \prod_{i=1}^{r} (1 - p^{-f_i s}) = \prod_{\mathfrak{q} | p\mathcal{O}_L} (1 - \mathrm{N}(\mathfrak{q})^{-s}).$$

We now show the equality

$$\prod_{\mathfrak{q} | p\mathcal{O}_L} (1 - \mathrm{N}(\mathfrak{q})^{-s}) = \prod_{\chi \in \widehat{G}} (1 - \chi^{\mathrm{prim}}(p) p^{-s}).$$

We make use of Theorem 6.13 in the lecture and write $H = \mathrm{Gal}(L/\mathbb{Q})$. Applied to our situation, the Theorem states that

$$\zeta_L(s) = \prod_{\mathfrak{q} | m'\mathcal{O}_L} \frac{1}{1 - \mathrm{N}(\mathfrak{q})^{-s}} \prod_{\chi \in \widehat{H}} L(s; \chi).$$

Hence (as $p \nmid m'$) the Euler factor of $\zeta_L(s)$ at p is given by

$$\prod_{\mathfrak{q} | p\mathcal{O}_L} \frac{1}{1 - \mathrm{N}(\mathfrak{q})^{s}} = \prod_{\chi \in \widehat{H}} \frac{1}{1 - \chi(p) p^{s}}.$$

Now the argument gets a little wild[1]. We need to show that

$$\prod_{\chi \in \widehat{H}} (1 - \chi(p) p^{-s}) = \prod_{\chi \in \widehat{G}} (1 - \chi^{\mathrm{prim}}(p) p^{-s}),$$

i.e., we need to study the relations between the character groups of H and G. Using standard arguments in Galois theory, one can show that the diagram of abelian groups

$$\begin{array}{ccc} \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) & \twoheadrightarrow & G \\ \downarrow & & \downarrow \\ \mathrm{Gal}(\mathbb{Q}(\zeta_{m'})/\mathbb{Q}) & \twoheadrightarrow & H \end{array}$$

[1] I was too lazy to formulate out what the category-theoretic words mean. But all of the following can be made explicit, and I encourage you to do so if you haven't encountered the words used here before.

is a pushout diagram in the category of (finitely generated) abelian groups. Now we use that $\mathrm{Hom}_{\mathsf{finAb}}(-, \mathbb{C}^\times)$ is an exact functor (or equivalently, $\mathbb{Q}/\mathbb{Z}$ is an injective object in Ab), to obtain that the dual diagram

$$\begin{array}{ccc} \widehat{H} & \hookrightarrow & \widehat{(\mathbb{Z}/m'\mathbb{Z})^\times} \\ \downarrow & & \downarrow \\ \widehat{G} & \hookrightarrow & \widehat{(\mathbb{Z}/m\mathbb{Z})^\times} \end{array}$$

is a pullback-diagram, which is to say, we have $\widehat{H} \cong \widehat{G} \cap \widehat{(\mathbb{Z}/m'\mathbb{Z})}^\times$ inside $\widehat{(\mathbb{Z}/m\mathbb{Z})}^\times$ (where we identify the multiplicative residue groups with the respective cyclotomic Galois-groups). Now we are almost done. It is relatively straight-forward to check that whenever $\chi \in \widehat{G}$, we have

$$\chi^{\mathrm{prim}}(p) = 0 \iff p \mid f_\chi \iff \chi \notin \widehat{(\mathbb{Z}/m'\mathbb{Z})}^\times \subset \widehat{(\mathbb{Z}/m\mathbb{Z})}^\times \iff \chi \notin \widehat{H}.$$

In particular, as we have $\chi(p) = \chi^{\mathrm{prim}}(p)$ whenever $p \nmid f_\chi$, we find

$$\prod_{\chi \in \widehat{H}} (1 - \chi(p)p^{-s}) = \prod_{\chi \in \widehat{G}} (1 - \chi^{\mathrm{prim}}(p)p^{-s}).$$

The claim follows.

3. This follows by comparing the factors of the respective Euler products.

Exercise 4

Let p be an odd prime number and let $\zeta \in \mathbb{C}$ be a root of unity of order p. Show that $1 + \zeta \in \mathcal{O}_{\mathbb{Q}(\zeta)}^\times$.

Solution. Note that $1 + \zeta = \frac{1-\zeta^2}{1-\zeta}$. If we pick $n \in \mathbb{N}$ such that $2n \equiv 1 \bmod p$, we have

$$\frac{1-\zeta^2}{1-\zeta} = \frac{1-\zeta^2}{1-\zeta^{2n}} = \left(\frac{1-\zeta^{2n}}{1-\zeta^2}\right)^{-1}.$$

And this is in $\mathcal{O}_K$, as

$$\frac{1-\zeta^{2n}}{1-\zeta^2} = 1 + \zeta^2 + \zeta^4 + \cdots + \zeta^{2(n-1)}.$$

Max von Consbruch, email: mvconsbruch@uni-bonn.de. Date: January 30, 2024