Approximate functional equation, what is it all about?!

1 A classical approximate functional equation

The aim of the approximate functional equation is to understand ζ (or more generally, any L-function) better in the critical strip $0 < \Re s < 1$. In this first part, we will focus on the case of ζ. When we proved that ζ has a meromorphic continuation to $\Re s > 0$, we used partial summation on the dirichlet series, showing that for $\Re s > 1$, we have

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{s}{s-1} - s\int_1^{\infty} \{t\} t^{-s-1}\,\mathrm{d}t,$$

where the RHS is defined also for $\Re s > 0$.

Given some $N > 0$, a similar expression arises if we use partial summation on the truncated Dirichlet series, as

$$\sum_{n \le N} n^{-s} = N^{1-s} + s\int_1^N \lfloor t \rfloor t^{-s-1}\,\mathrm{d}t = \frac{N^{1-s}}{1-s} + \frac{s}{s-1} - s\int_1^N \{t\} t^{-s-1}\,\mathrm{d}t.$$

Now one might be tempted to comare the RHSs of the previous two equations. Writing $s = \sigma + it$, we find

$$\zeta(s) - \sum_{n \le N} n^{-s} = \frac{N^{1-s}}{s-1} - s\int_1^N \{t\} t^{-s-1}\,\mathrm{d}t = \frac{N^{1-s}}{s-1} + O\left(\frac{|s|}{\sigma} N^{-\sigma}\right).$$

The important observation is that nothing goes wrong if we pass from $\Re s > 1$ to $\Re s > 0$! We found that ζ is approximated by the first terms in its dirichlet series, even in the critical strip. As is turns out, this approximation is not great, as we still have that annoying $|s|$ in the O-term, which forces us to choose N large (roughly like $t^{1/\sigma}$) to make use of this approximation. The crucial thing we missed in our approximation is that $n^{it} = \mathrm{e}^{(\log n)it}$ oscillates and constitutes a lot of cancellation. Using some sort of approximate fourier transform called *van der Corput summation*, one can get hold of this oscillation to obtain a stronger bound on the error, given by

$$\zeta(s) = \sum_{n \le x} n^{-s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}).$$

This is uniform in $\sigma > \sigma_0$ once we fix $\sigma_0 > 0$, provided that $|t| \le 4x$. (Take a look in Chapter 4 of Brüdern's book for details).

Choosing $\sigma = \frac{1}{2}$, we find that

$$\zeta(s) \ll \sum_{n \le t} n^{-\sigma} + t^{1-\sigma} \ll t^{1-\sigma},$$

which is an okay bound, but not as good as we'd like. The convexity bound already gave that $\zeta(s) \ll t^{\frac{1-\sigma}{2}+\varepsilon}$, so we could hope that we could do even better, approximating ζ with sums of length $\sqrt{t}$. Unfortunately, it does not seem as if such a identity holds true.

However, we can apply the functional equation to obtain a similar approximation of ζ, just from the other side (i.e., at $1-s$). Even better, we might be able to combine these approximations to obtain a better approximation of ζ. This is the Idea of the *approximate functional equation.* And indeed, it gives what we hoped for: We can essentially approximate ζ by Dirichlet-sums of length $\sqrt{t}$. If we write $\zeta(s) = \Delta(s)\zeta(1-s)$, the theorem reads as

Theorem 1.1 (Approximate functional equation)**.** *Let* $0 < \sigma < 1$ *and* $2\pi xy = t$ *, where* $x, y > 1$*. Then*

$$\zeta(s) = \sum_{n\le x} n^{-s} + \Delta(s)\sum_{n\le y} n^{s-1} + O((x^{-\sigma} + t^{1/2-\sigma}y^{\sigma-1})\log t).$$

We shouldn't worry about the shape of the error term too much, just observe that the balanced case is given when $\sigma = \frac{1}{2}$ and $x = y = \sqrt{\frac{x}{2\pi}}$.

This AFE is stronger than the one we had in the lecture. For example, it allows us to deduce asymptotic formulas for the second and fourth moments of ζ on the critical line, only using elementary manipulations. We get

$$\int_0^T \left|\zeta(\tfrac{1}{2} + \mathrm{i}t)\right|^2 \,\mathrm{d}t = T\log T + O(T)$$

and

$$\int_0^T \left|\zeta(\tfrac{1}{2} + \mathrm{i}t)\right|^4 \,\mathrm{d}t = \frac{T(\log T)^4}{2\pi} + O(T(\log T)^3).$$

(Again, you can read this up in Brüdern's book, chapter 4).

2 The smoothed approximate functional equation

Okay, the approximate functional equation is cool (or not), but how does it relate to the approximate functional equation we had in the lecture?! It looked much more complicated and did not allow us to deduce asymptotic formulas. In the lecture, we proved a smoothed version of the formula above, and usually, smoothed formulas are easier to prove, but harder to use. (One reason for this phenomenon is that the mellin tranform transforms smoothness into rapid decay along vertical lines. This usually makes calculations easy.)

We are given some Dirichlet L-function $L(s) = \sum_{n=1}^\infty a_n n^{-s}$, which we want to approximate by a sum of the form $\sum_{n=1}^\infty a_n n^{-s}\omega(n/X)$, where $X > 0$ and $\omega = \omega_s$ is some smooth cut-off function. Denoting the Mellin transform of ω by $\hat{\omega}$, we obtain (similarly to what we did for Perron's formula)

$$\sum_{n=1}^\infty a_n n^{-s}\omega\left(\frac{n}{X}\right) = \frac{1}{2\pi\mathrm{i}}\int_{(c)} L(s+u)X^u\hat{\omega}(u)\,\mathrm{d}u. \tag{2.1}$$

Now, as we always do, we hope to find something new when shifting the contour of the integral to the left, to $(-c)$, say. We now that we want $L(s)$ to appear, so it seems advisable to choose ω such that $\hat{\omega}$ has a simple pole with residue 1 at zero. We also might pick up residues of L at $u = -s$ and $u = 1-s$, we will denote these residues with R. We get

$$\sum_{n=1}^\infty a_n n^{-s}\omega(n/X) = L(s) + \frac{1}{2\pi\mathrm{i}}\int_{(-c)} L(s+u)\hat{\omega}(u)X^u\,\mathrm{d}u + R.$$

What now? Usually we would like to bound the integral, but showing that this integral is small seems impossible. However, as we understand $L(s)$ way better at $\Re s = c$ than at $\Re s = -c$, we could try to apply the functional equation. The functional equation for L reads

$$\Lambda(s) = \eta\overline{\Lambda}(1-s), \quad \text{where} \quad \Lambda(s) = N^{s/2}L_\infty(s)L(s).$$

Hence we write $\hat{\omega}(u) = \frac{L_\infty(s+u)G(u)}{sL_\infty(s)}$, and we impose that G is a function independent of u with $G(0) = 1$. We obtain

$$\sum_{n=1}^{\infty} a_n n^{-s} \omega(n/X) = L(s) + \frac{\eta}{N^{s/2} L_\infty(s)} \cdot \frac{1}{2\pi \mathrm{i}} \int_{(-c)} \Lambda(s+u) G(u) \left(\frac{X}{\sqrt{N}}\right)^u \frac{\mathrm{d}u}{u} + R \tag{2.2}$$

Investigating the integral further, we find after applying the functional equation and a few manipulations

$$\begin{aligned}
\frac{1}{2\pi \mathrm{i}} \int_{(-c)} \Lambda(s+u) G(u) \left(\frac{X}{\sqrt{N}}\right)^u \frac{\mathrm{d}u}{u} &= \frac{1}{2\pi \mathrm{i}} \int_{(-c)} \Lambda(1-s-u) G(u) \left(\frac{X}{\sqrt{N}}\right)^u \frac{\mathrm{d}u}{u} \\
&= -\frac{1}{2\pi \mathrm{i}} \int_{(c)} \overline{\Lambda}(1-s+u) G(-u) \left(\frac{\sqrt{N}}{X}\right)^u \frac{\mathrm{d}u}{u} \\
&= -\frac{1}{2\pi \mathrm{i}} \int_{(c)} \overline{\Lambda}(1-s+u) G(-u) \left(\frac{\sqrt{N}}{X}\right)^u \frac{\mathrm{d}u}{u}.
\end{aligned}$$

If we further impose that G is an even function (i.e. $G(u) = G(-u)$), this is almost the integral we started with in (2.1)! We can reverse our initial procedure and obtain

$$\begin{aligned}
&\frac{1}{2\pi \mathrm{i}} \int_{(c)} \overline{\Lambda}(1-s+u) G(-u) \left(\frac{\sqrt{N}}{X}\right)^u \frac{\mathrm{d}u}{u} \\
&= \frac{N^{\frac{1-s}{2}} L_\infty(1-s)}{2\pi \mathrm{i}} \int_{(c)} \overline{L}(1-s+u) \hat{\omega}_{1-s}(u) \left(\frac{N}{X}\right)^u \mathrm{d}u = N^{\frac{1-s}{2}} L_\infty(1-s) \sum_{n=1}^{\infty} \frac{\overline{a_n}}{n^{1-s}} \omega_{1-s}\left(\frac{nX}{N}\right).
\end{aligned}$$

Plugging this into (2.2), we get

$$L(s) = \sum_{n=1}^{\infty} a_n n^{-s} \omega_s\left(\frac{n}{X}\right) + \eta N^{1/2-s} \frac{L_\infty(1-s)}{L_\infty(s)} \sum_{n=1}^{\infty} \frac{\overline{a_n}}{n^{1-s}} \omega_{1-s}\left(\frac{nX}{N}\right) - R, \tag{2.3}$$

which is exactly the statement of the approximate functional equation of the lecture once we replace ω by the inverse mellin transform of $\hat{\omega}$, and X by $X\sqrt{N}$.

TL;DR: The smoothed approximate functional equation we had in the lecture is just another instance of a recurrent procedure: We start with some (possibly smoothly) weighted sum $\sum_{n\in\mathbb{N}} a_n \omega(n/x)$, calculate the mellin transform $\hat{\omega}$ of ω and obtain for $x > 0$

$$\sum_{n\in\mathbb{N}} a_n \omega(n/x) = \frac{1}{2\pi \mathrm{i}} \int_{(c)} F(s) \hat{\omega}(s) x^s \,\mathrm{d}s.$$

We now shift the contour to the left, pick up residues and make up methods to deal with the remaining integral. In our case, we really care more about what's happening in the integrals than in the sum, so we first choose a function we want to have as mellin transform and then choose ω in a way such that it mellin transforms into that function. (This leads to the awkward definition of V_s). With this choice of a smooth weight we are able to pick up $L(s)$ as a residue (which is what we wanted to approximate) and use the functional equation for L-functions to reinterpret the shifted contour as a similar smoothly weighted sum again. This leads to (2.3).