Solution to Sheet 4.

Problem 1

a) Let $g(x) = f(qx+a)$, so that

$$\sum_{n\equiv a \pmod q} f(n) = \sum_{m\in\mathbb{Z}} g(m).$$

We want to apply Poisson summation to g. The results of lemma (2.3) directly give that

$$\hat{g}(y) = \frac{1}{q} e\left(\frac{ya}{q}\right) \hat{f}\left(\frac{y}{q}\right).$$

The claim follows, as

$$\sum_{m\in\mathbb{Z}} g(m) = \sum_{m\in\mathbb{Z}} \hat{g}(m) = \frac{1}{q}\sum_{m\in\mathbb{Z}} e\left(\frac{ma}{q}\right) \hat{f}\left(\frac{m}{q}\right).$$

b) We would like to apply Poisson summation again, however we cannot calculate the "Fourier transform" of $f\chi$, as, χ is only defined on integers. We can abuse that χ is periodic though, rewriting

$$\sum_{m\in\mathbb{Z}} f(m)\chi(m) = \sum_{a \pmod q} \chi(a) \sum_{m\equiv a \pmod q} f(m).$$

Applying Poisson summation to the inner sum (we already did this in part a)) gives

$$\sum_{m\in\mathbb{Z}} f(m)\chi(m) = \frac{1}{q}\sum_{a \pmod q} \chi(a) \sum_{m\in\mathbb{Z}} e\left(\frac{ma}{q}\right) \hat{f}\left(\frac{m}{q}\right).$$

Reordering sums, we obtain

$$\begin{aligned}\frac{1}{q}\sum_{a \pmod q} \chi(a) \sum_{m\in\mathbb{Z}} e\left(\frac{ma}{q}\right) \hat{f}\left(\frac{m}{q}\right) &= \frac{1}{q}\sum_{m\in\mathbb{Z}} \hat{f}\left(\frac{m}{q}\right)\left(\sum_{a \pmod q}\chi(a) e\left(\frac{ma}{q}\right)\right) \\ &= \frac{1}{q}\sum_{m\in\mathbb{Z}} \hat{f}\left(\frac{m}{q}\right)\tau(\chi)\overline{\chi}(m) = \frac{\tau(\chi)}{q}\sum_{m\in\mathbb{Z}} \hat{f}\left(\frac{m}{q}\right)\overline{\chi}(m).\end{aligned}$$

Notes after correcting.

- In part a), instead of using the results from the lecture, we can also obtain the formula for the fourier transform directly. Setting $g(x) = f(qx+a)$ and substituting $u = qx+a$, we obtain

$$\hat{g}(y) = \int_{\mathbb{R}} f(qx+a)e(-xy)\,\mathrm{d}x = \frac{1}{q}\int_{\mathbb{R}} f(u)e(-u\tfrac{y}{q} + \tfrac{ay}{q})\,\mathrm{d}u = \frac{1}{q}e(\tfrac{ay}{q})\hat{f}(\tfrac{y}{q}).$$

Problem 2

We do as the hint commands. Let

$$f(t) = \begin{cases} \mathrm{e}^{-1/t^2} & t > 0 \\ 0 & \text{else.} \end{cases}$$

Then one easily checks that f is smooth and non-negative. Now we put $g(t) = \frac{f(t)}{f(t)+f(1-t)}$, which is still smooth and non-negative. We clearly have $g(t) = 0$ if $t < 0$, $g(t) \in [0,1]$ for $t \in [0,1]$ and $g(t) = 1$ for $t > 1$. Finally, define

$$h(t) = g\left(\frac{t - X + Z}{Z}\right) - g\left(\frac{t - X - Y}{Z}\right).$$

This satisfies $\mathrm{supp}(h) \subset [X - Z, X + Y + Z]$ and $h(t) = 1$ for $t \in [X, X + Y]$. We still need to check that $\|f^{(j)}\|_1 \ll Z^{1-j}$ for all $j \in \mathbb{N}$. One could expect this to be really messy as calculating the higher derivatives of h seems horrible. However, we just need that the j-th derivative of h is given by

$$h^{(j)}(t) = Z^{-j}\left(g^{(j)}\left(\tfrac{t-X+Z}{Z}\right) - g^{(j)}\left(\tfrac{t-X-Y}{Z}\right)\right).$$

As $h^{(j)}$ vanishes everywhere except $[X - Z, X]$ and $[X + Y, X + Y + Z]$, we obtain by a linear change of variables

$$\|h^{(j)}\|_1 = \left(\int_{X+Z}^{X} + \int_{X+Y}^{X+Y+Z}\right)\left|h^{(j)}(t)\right| \,\mathrm{d}t = 2Z^{1-j}\int_0^1 \left|g^{(j)}(t)\right| \,\mathrm{d}t \ll_j Z^{1-j}.$$

Problem 3

As the hint commands, we apply partial summation to the definition of $\tau(\chi)$, obtaining

$$|\tau(\chi)| = \sum_{h=1}^{q} \chi(h)e(h/q) = e(q/q)\sum_{h=1}^{q} \chi(h) - \frac{2\pi \mathrm{i}}{q}\int_1^q e(t/q)\sum_{h \le t} \chi(h)\,\mathrm{d}t.$$

As $\chi \neq \chi_0$, the sum $\sum_{h=1}^{q} \chi(h)$ vanishes. We also know by theorem (1.23) that $|\tau(\chi)| = \sqrt{q}$. Let M deonte the supremum of the absolute values of $\sum_{h \le x} \chi(h)$ for varying x (By Polya-Vinogradov, $M < \infty$). Then we obtain

$$\frac{q^{3/2}}{2\pi} = \left|\int_1^q e(t/q)\sum_{h\le t} \chi(h)\,\mathrm{d}t\right| \le \int_1^q \left|\sum_{h \le t} \chi(h)\right| \,\mathrm{d}t \le (q-1)M,$$

which is even a tad stronger than what we had to show.

Problem 4

Let's just plug in the definition and look at what we have here.

$$\tau(\chi_1\chi_2) = \sum_{h\ (q)} \chi_1(h)\chi_2(h)e(h/q),$$

where $q = q_1 q_2$. By the chinese remainder theorem, taking residues mod q gives a bijection

$$\{h_1 q_2 + h_2 q_1 \mid 1 \le h_i \le q_i\} \to \mathbb{Z}/q\mathbb{Z}.$$

Thus we may rewrite the sum above as

$$\tau(\chi_1\chi_2) = \sum_{1\le h_1\le q_1} \sum_{1\le h_2\le q_2} \chi_1(h_1q_2 + h_2q_1)\chi_2(h_1q_2 + h_2q_1)e(\tfrac{h_1q_2+h_2q_1}{q}),$$

and the claim follows after a few manipulations:

$$\begin{aligned}
&\sum_{1\le h_1\le q_1} \sum_{1\le h_2\le q_2} \chi_1(h_1q_2 + h_2q_1)\chi_2(h_1q_2 + h_2q_1)e(\tfrac{h_1q_2+h_2q_1}{q}) \\
&\qquad = \sum_{1\le h_1\le q_1} \sum_{1\le h_2\le q_2} \chi_1(h_1q_2)\chi_2(h_2q_1)e(\tfrac{h_1q_2}{q})e(\tfrac{h_2q_1}{q}) \\
&= \left(\chi_1(q_2) \sum_{1\le h_1\le q_1} \chi_1(q_2)e(\tfrac{h_1}{q_1})\right)\left(\chi_2(q_1) \sum_{1\le h_2\le q_2} \chi_2(q_1)e(\tfrac{h_2}{q_2})\right) = \chi_1(q_2)\tau(\chi_1)\chi_2(q_1)\tau(\chi_2).
\end{aligned}$$

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