Solutions to Sheet 7.

Problem 1

a - 2p) We have $g(x) \ll x^{-(-u+av)}$ as $x \to 0$ and $g(x) \ll x^{-(-u+bv)}$ as $x \to \infty$. Hence in $-u + av < \mathrm{Re}(s) < -u + bv$ the mellin transform $\widehat{g}$ exists and is given by

$$\widehat{g}(s) = \int_0^\infty x^u f(x^v) x^s \frac{\mathrm{d}x}{x} = v^{-1} \int_0^\infty f(y) y^{(s+u)/v-1} \,\mathrm{d}y = v^{-1} f\left(\frac{s+u}{v}\right)$$

Now the RHS defines a holomorphic function in $-u + a'v < \mathrm{Re}(s) < -u + b'v$.

b - 3p) Of course, knowing bounds for f does not imply any bounds for f'. But knowing that we can derive f, we can make use of partial integration. We have

$$\int_0^\infty f(x) x^{s-1} \,\mathrm{d}x = \left[f(x) \frac{x^s}{s} \right]_0^\infty - \frac{1}{s} \int_0^\infty f'(x) x^s \,\mathrm{d}x$$

By assumption, the boundary terms vanish for $a < \mathrm{Re}(s) < b$, and the integral on the RHS exists (if this is not clear, try to first approximate the integrals by truncated ones from $1/T$ to T and let $T \to \infty$). Hence $\widehat{g}$ (with $g = f'$) exists in $a + 1 < \mathrm{Re}(s) < b + 1$ (note the shift $s \mapsto s + 1$ in the integral). Same argument as before gives continuation of $\widehat{g}$ to $a' + 1 < \mathrm{Re}\, s < b' + 1$.

c - 3p) By assumption f has compact support, so the Mellin Transform exists everywhere and the same holds for the derivatives. We make use of what we showed in *b*) repeatedly, obtaining

$$\widehat{f}(s) = \frac{(-1)^N}{s(s+1)\dots(s+N-1)} \widehat{g}(s+N) = (-1)^N \frac{\Gamma(s)}{\Gamma(s+N)} \widehat{g}(s+N)$$

where $g = f^{(N)}$. The first Γ-factor behaves (for fixed real part and large imaginary part of s) like $O(|s|^{-N})$, so it remains to show that $\widehat{g}(s)$ is bounded with $\mathrm{Im}\, s \to \infty$. But the integral from the mellin transform can be bounded in absolute values, as

$$|g(s)| \leq \int_0^\infty \left| g(x) x^{s-1} \right| \mathrm{d}x \ll \int |g(x)|\, x^{\mathrm{Re}(s)-1} \,\mathrm{d}x.$$

This is convergent, and independent of $\mathrm{Im}(s)$.

d - 2p) Calculation:

$$\begin{aligned}\widehat{f \star h}(s) = \int_0^\infty (f \star h)(x) x^{s-1} \,\mathrm{d}x = \int_0^\infty \int_0^\infty f(t) h(x/t) t^{-1} \,\mathrm{d}t x^{s-1} \,\mathrm{d}x \\ = \int_0^\infty f(t) h(y) t^{s-1} y^{s-1} \,\mathrm{d}t \,\mathrm{d}y,\end{aligned}$$

as desired. We made use of the substitution $y = x/t$, i.e. $\mathrm{d}y = t^{-1} \,\mathrm{d}x$.

Problem 2&3

a - 15p) We want to apply Perron. Remember that we showed earlier that the Dirichlet series attached to the characteristic function on the set of squarefull numbers is given by $\frac{\zeta(2s)\zeta(3s)}{\zeta(6s)}$.

Just as in one of the examples from the lecture, we apply Perron with $c = 1 + 1/\log x$ and $T = x^\alpha$ for some fixed $\alpha \in (0,1)$. The absolute value of the coefficients is ≤ 1 and we obtain

$$\sum_{n \le x \text{ sqfull}} 1 = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}T}^{c+\mathrm{i}T} \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)} x^s \frac{\mathrm{d}s}{s} + O(T^{-1} x \log x).$$

We want to shift the contour to the left and pick up residues along the way. The most important tool to bound the vertical contribution is the moment bound, and this requires the real part of the argument to be at least $\frac{1}{2}$. Hence we shift to $\operatorname{Re} s = \frac{1}{4}$. The factor $\zeta^{-1}(6s)$ is still holomorphic here, so we only pick up the residues from $\zeta(2s)$ and $\zeta(3s)$. We obtain

$$\sum_{n \le x \text{ sqfull}} 1 =$$

$$\frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + \left(\int_{c-\mathrm{i}T}^{1/4-\mathrm{i}T} + \int_{1/4-\mathrm{i}T}^{1/4+\mathrm{i}T} + \int_{1/4+\mathrm{i}T}^{c+\mathrm{i}T} \right) \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)} x^s \frac{\mathrm{d}s}{s} + O(T^{-1} x \log x).$$

First, note that $\zeta^{-1}(s)$ is bounded in $\operatorname{Re} s > 1 + \delta$, as

$$\left| \zeta^{-1}(s) \right| = \prod_p |1 - p^{-s}| \le \prod_p (1 + p^{-1-\delta}) = \frac{\zeta(2+2\delta)}{\zeta(1+\delta)} \ll_\delta 1.$$

So we disregard this factor from now on. Let us first start with the vertical part. Here we have $|x^s| = x^{1/4}$, so the contribution is bounded by

$$\ll x^{1/4} \int_0^T \frac{|\zeta(1/2 + 2\mathrm{i}t)\zeta(3/4 + 3\mathrm{i}t)|}{1/4 + \mathrm{i}t} \mathrm{d}t.$$

We prove that the integral is bounded by x^ε. By splitting the integral into $\log x$ dyadic pieces $[M, 2M]$ for $M < T$. It suffices to show that

$$\int_M^{2M} \frac{|\zeta(1/2 + 2\mathrm{i}t)\zeta(3/4 + 3\mathrm{i}t)|}{1/4 + \mathrm{i}t} \mathrm{d}t \ll M^{1+\varepsilon}.$$

The denumerator is (throughout) of size $\gg M$, so we really only need to show that

$$\int_M^{2M} |\zeta(1/2 + 2\mathrm{i}t)\zeta(3/4 + 3\mathrm{i}t)| \, \mathrm{d}t \ll M^\varepsilon \ll T^\varepsilon$$

This is an immediate consequence of Cauchy-Schwartz and the moment bounds. Hence we can conclude

$$\begin{aligned} &\int_0^T \frac{|\zeta(1/2 + 2\mathrm{i}t)\zeta(3/4 + 3\mathrm{i}t)|}{1/4 + \mathrm{i}t} \mathrm{d}t \\ &\qquad \le \left(\int_0^1 + \int_1^2 + \cdots + \int_{2^{\lfloor \log_2(T) \rfloor}}^{2^{\lfloor \log_2(T) \rfloor + 1}} \right) \frac{|\zeta(1/2 + 2\mathrm{i}t)\zeta(3/4 + 3\mathrm{i}t)|}{1/4 + \mathrm{i}t} \mathrm{d}t \ll \log_2(T) T^\varepsilon \ll T^\varepsilon. \end{aligned}$$

Next, we focus on the horizontal parts. Here, $s^{-1} \ll T^{-1}$, so the contributions become

$$\ll T^{-1} \int_{1/4}^{c} |\zeta(2(\sigma + \mathrm{i}T))\zeta(3(\sigma + \mathrm{i}T)| \, \mathrm{d}\sigma \ll T^{-1} \int_{1/4}^{c} T^{\max(1/2-\sigma,0)} T^{\max(1/2-3\sigma/2,0)} x^\sigma \, \mathrm{d}\sigma.$$

This requires some bookkeeping, but splitting this into the parts $(1/4, 1/3)$, $(1/3, 1/2)$ and $(1/2, c)$ one quickly verifies that no term contriibutes more that $x^{1+\varepsilon}$. To this end, we showed

$$\sum_{n \le x \text{ sqfull}} 1 = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + O\left(\frac{x^{1+\varepsilon}}{T} + x^{1/4+\varepsilon} \right).$$

The claim follows upon setting $T = x^{3/4}$.

b - 5p) The good thing with smooth weights is that their mellin transforms usually decay quickly along vertical lines and we do not have to worry about cutting off the integral. Perron's formula reveals with $c > 1/2$

$$\sum_{n \text{ squarefull}} \mathrm{e}^{-n/x} = \frac{1}{2\pi \mathrm{i}} \int_{(c)} \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)} x^s \Gamma(s) \, \mathrm{d}s.$$

As Γ vanishes rapidly along vertical lines, we can shift the contour to $\operatorname{Re} s = 1/6 + \varepsilon$ and obtain

$$\cdots = \frac{1}{2} \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{1}{3} \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + \frac{1}{2\pi \mathrm{i}} \int_{(1/6+\varepsilon)} \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)} x^s \Gamma(s) \, \mathrm{d}s.$$

The integral is absolutely convergent, hence gives an error of size $O(x^{1/6+\varepsilon})$.

Remark: We will later prove that $\zeta(s)$ does not have zeroes in some neighbourhood of the line $\operatorname{Re} s = 1$, which in particular implies that there are no zeroes on the line itself. Hence we can get even shift the contour onto $\operatorname{Re} s = 1/6$, killing the $+\varepsilon$.

Problem 4

a - 6p) Every finite abelian group can be decomposed as a product of cyclic groups of prime-power-order. Hence the number of isomorphism classes of abelian groups of order n gives a multiplicative arithmetic function

$$a : \mathbb{N} \to \mathbb{N}, \quad n \mapsto \#(\{\text{abelian groups of order } n\}/\cong).$$

If $n = p^r$ is a prime power, we find that $a(n)$ is given by the number of (additive) partitions of r. Indeed, to a partition

$$1 \cdot a_1 + 2 \cdot a_2 + 3 \cdot a_3 + \cdots = r$$

we can associate a group $(\mathbb{Z}/p\mathbb{Z})^{a_1} \times (\mathbb{Z}/p^2\mathbb{Z})^{a_2} \times (\mathbb{Z}/p^3\mathbb{Z})^{a_3} \times \ldots$ of order p^r, and vice versa. One quickly verifies (at least formally), that

$$\sum_{n=1}^{\infty} a(n) x^n = (1 + x + x^2 + \ldots)(1 + x^2 + x^4 + \ldots)(1 + x^3 + x^6 + \ldots) \cdots$$

and substituting $x = p^{-s}$ for varying p yields the desired formula

$$\sum_{n=1}^{\infty} a(n) n^{-s} = \prod_p \prod_{r=1}^{\infty} (1 - p^{-rs})^{-1} = \prod_{r=1}^{\infty} \zeta(rs).$$

The last step might demand clearification. Remember that a product $\prod a_n$ with $a_n \neq 0$ converges absolutely to something $\neq 0$ iff the sum $\sum |a_n - 1|$ converges absolutely. In $\operatorname{Re} s > 1 + \delta$ we have the uniform bound

$$|1 - \zeta(rs)| \ll \sum_{n=2}^{\infty} n^{r(-1-\delta)} \ll_\delta 2^{-r},$$

so that which shows that indeed, the product converges absolutely and locally uniformly in $\operatorname{Re} s > 1$.

b - 4p) The heuristic goes as follows. Let F be the Dirichlet series attached to a. By the above, F is a holomorphic function for $s > 1$, but by the continuation of the first ζ-factor, we find that F has a continuation to a meromorphic function on $\mathrm{Re}\, s > 1/2$. (Aside: We can apply the functional equation to as many ζ-factors as we want, yielding continuations to $\mathrm{Re}\, s > 1/n$ for arbitrarily large $n \in \mathbb{N}$. But F can never be meromorphically continued to all of $\mathbb{C}$. This is because there are poles at $s = 1, 1/2, 1/3, \ldots$, which by the identity theorem implies that $F^{-1} = 0$.) Now Perron's Formula reads

$$\sum_{n \leq x} a(n) = \frac{1}{2\pi \mathrm{i}} \int_{(c)} F(s) x^s \frac{\mathrm{d}s}{s},$$

and upon shifting the contour to $1 - \varepsilon$ we obtain

$$\sum_{n \leq x} a(n) = x \mathrm{Res}_{s=1} F(s) + \frac{1}{2\pi \mathrm{i}} \int_{(1-\varepsilon)} F(s) x^s \frac{\mathrm{d}s}{s}.$$

The residue is given by $C = \zeta(2)\zeta(3) \cdots$, and we'd hope that we would be able to approximate the integral by something of size $o(x)$.

Proving the asymptotic. Proving the asymptotic is quite challenging, as we would have to find some bound on $a(n)$ to apply (4.7). The convergence of $\sum_n a(n) n^{-s}$ for $\mathrm{Re}(s) > 1$ gives $a(n) \ll n^{1+\varepsilon}$, but there is no trivial way to get anything beyond that. But it turns out we don't need such bounds! Note that we really need to include a bound of $a(n)$ in (4.7) because we try to approximate a function that "jumps" (the LHS) with a function that is continuous in x (the integral, at least as long as $T = T(x)$ is continuous in x). But if we decide to inspect the approximation away from the jumps of the LHS, we might be able to prove an error not involving terms of the form $O(\max_{n \sim x} |a_n|)$. This idea is sketched in the following.

Using a modified version of (4.7). The probably more sensible way to do this is to use a modified version of (4.7): If we assume $x \in \frac{1}{2} + \mathbb{N}$ (more generally, $x \in [\delta, 1 - \delta] + \mathbb{N}$ works for $0 < \delta < 1/2$), we can copy the proof of (4.7), but the first summand A_x can be avoided. This gives (with the same terminology as in (4.7)) the statement

$$\sum_{n \leq x} a_n = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}T}^{c+\mathrm{i}T} \sum_{n \in \mathbb{N}} \frac{a_n}{n^s} x^s \frac{\mathrm{d}s}{s} + O\left(\frac{x^c}{T} \sum_n \frac{|a_n|}{n^c} + A_x \frac{x \log x}{T} \right).$$

We can now follow the same strategy as usual, and in the end realize that $T = x^\alpha$ can be chosen an arbitrary power of x, which should ultimately yield a asymptotic with error $O(x^{1/2+\varepsilon})$. (You will need $A_x = \max_{n \sim x} |a_n| \ll x^{1+\varepsilon}$.) This is left as an exercise :)

The following solution introduces a new idea. We sacrifice a bit of error size, but get a smooth ride when moving the integral to the left in exchange. You will realize we almost don't have to worry about messy calculations at all!

Proving the asymptotic using Cesàro-weights. Instead of trying to avoid the jumps, we could also try to smooth out the LHS of (4.7). Instead of bounding

$$S_0(x) = \sum_{n \leq x} a(n),$$

we try to bound

$$S_1(x) = \sum_{n \leq x} a(n)(x - n) = \int_1^x S_0(y)\, \mathrm{d}y.$$

(These weights are called Cesàro weights). We hope to recover information about S_0 afterwards. Integrating Perron's formula, we find that

$$S_1(x) = \frac{1}{2\pi \mathrm{i}} \int_{(c)} F(s) x^{s+1} \frac{\Gamma(s)}{\Gamma(s+2)} \,\mathrm{d}s.$$

The Γ-factor is essentially bounded by $|s|^{-2}$, at least for $|s| > 2$. Whenever $\sigma > 1/2 + \delta$ and $|t| > 1$ we find

$$F(\sigma + \mathrm{i}t) \ll |\zeta(s)|\, \zeta(1 + 2\delta)\zeta(3/2 + 3/2\delta) \cdots \ll |t|^{\frac{1-\sigma}{2} + \varepsilon} \, \delta^{-1}.$$

Hence we can shift the contour to $\operatorname{Re} s = 1/2 + \delta$, pick up a pole and the remaining integral remains absolutely convergent. In formulas,

$$S_1(x) = \frac{x^2}{2} C + \int_{(1/2+\delta)} F(s) x^{s+1} \frac{\Gamma(s)}{\Gamma(s+2)} \,\mathrm{d}s = \frac{x^2}{2} C + O_\delta(x^{3/2+\delta}).$$

Nice, this at least shows that there is an asymptotic *on average*. But how can we make use of this? We also showed that the Lindelöf-Hypothesis is true *on average*, but we are far from proving the Lindelöf-Hypothesis in general! What plays in our favor here is that S_0 is non-decreasing. Denote by $E_0(x)$ the error function $S_0(x) - Cx$, and define E_1 as the integral of E_0. Note that we have $E_1(x) \ll x^{3/2+\varepsilon}$ by the above. We also make a choice of some $Q = x^\alpha$ for $\alpha \in [0, 1]$ and get (using monotonicity of S_0)

$$E_1(x + Q) - E_1(x) = \int_x^{x+Q} E_0(t) \,\mathrm{d}t \geq Q(S_0(x) - Cx - CQ) = QE_0(x) + O(Q^2).$$

But we also know that $E_1(x + Q) - E_1(x) = O(x^{3/2+\varepsilon})$, implying

$$QE_0(x) \leq O(x^{3/2+\varepsilon} + Q^2).$$

This shows $E_0(x) \leq O(x^{3/4+\varepsilon})$ once we choose $Q = x^{3/4}$. A similar lower bound can be established by inspecting $\int_{x-Q}^{x} E_0(t) \,\mathrm{d}t$ (exercise, haha). This proves $S_0(x) = Cx + O(x^{3/4+\varepsilon})$. This really is remarkable, as this in particular implies that $a(n) \ll n^{3/4+\varepsilon}$, which is a bound we did not know existed beforehand. Even more, this followed only from a bound on the vertical growth of $F(s)$ and the fact that $a(n) \geq 0$. Also note that we by did not do as good as we could have! We could have shifted further to the left and picked up more residues.

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