

# Solutions to Sheet 8.

## Problem 1

**a-4p)** Look at sheet 5.

**b-6p)** The approximation in (4.9) reads

$$\sum_{n \leq x} d_k(n) = xP_k(\log x) + O(x^{1-\delta}).$$

Reading the proof reveals that the main term is given by the residue

$$R := \operatorname{Res}_{s=1} \frac{\zeta^k(s)x^s}{s} = \operatorname{Res}_{s=1} F_k(s),$$

where for convenience  $F_k(s) := \frac{\zeta^k(s)x^s}{s}$ . Of course  $\zeta^k(s)$  has a singularity of degree  $k$  at 1, and we only need to calculate the  $(-1)$ st term of the Laurent expansion of  $F$  at 1. We have the Taylor expansions

$$\frac{1}{s} = \sum_{n=0}^{\infty} (-1)^n (s-1)^n = 1 - (s-1) + (s-1)^2 + O((s-1)^3)$$

and

$$x^s = \sum_{n=0}^{\infty} \frac{x(\log x)^n}{n!} (s-1)^n = x + x(\log x)(s-1) + \frac{1}{2}x(\log x)^2(s-1)^2 + O((s-1)^3).$$

Let  $a_n$  denote the coefficients of the Laurent series of  $\zeta$  at 1, i.e.

$$\zeta(s) = \sum_{n=-1}^{\infty} a_n (s-1)^n.$$

Calculating  $P_2$  and  $P_3$  now is pure calculation.

**Calculating  $P_2$ .** We find

$$\zeta^2(s) = \left( \sum_{n=-1}^{\infty} a_n s^n \right)^2 = a_{-1}^2 (s-1)^{-2} + 2a_{-1}a_0 (s-1)^{-1} + O(1).$$

We multiply this with the Taylor series above and find that the coefficient of  $(s-1)^{-1}$  is given by

$$2a_{-1}a_0x + a_{-1}^2(x \log x - x) = x(a_{-1}^2 \log x + 2a_{-1}a_0 - a_{-1}^2).$$

**Remark.** It is possible to show by elementary means that

$$\sum_{n \leq x} d_2(n) = x \log x + (2\gamma - 1)x + O(x^{1/2}),$$

which shows that  $a_0 = \gamma$ . (I just realized that  $a_{-1} = 1$  is already known haha, but we could probably also derive this with a similar approach and  $k = 1$ ). This shows

$$P_2(X) = X + 2a_0 - 1.$$

**Calculating  $P_3$ .** We find similarly to above

$$\zeta^3(s) = (s-1)^{-3} + 3a_0(s-1)^{-2} + 3(a_0^2 + a_1)(s-1)^{-1} + O(1).$$

and again use this to figure out the coefficient of  $(s-1)^{-1}$  in the Laurent expansion of  $F_3$  around  $s = 1$ . We find that this coefficient is given by

$$\begin{aligned} 3(a_0^2 + a_1)x + 3a_0x(\log x - 1) + x((\log x)^2 - \log x + 1) \\ = x\left(\frac{1}{2}(\log x)^2 + (\log x)(3\gamma - 1) + 3(\gamma^2 + a_1 - \gamma) + 1\right), \end{aligned}$$

i.e.

$$P_3(X) = \frac{1}{2}x^2 + (3\gamma - 1)x + 3(a_1 + a_0^2 - a_0) + 1.$$

## Problem 2

a) To calculate  $\zeta'(0)$ , we make use of the functional equation, in the form

$$\zeta(1-s) = \zeta(s) \cdot \frac{2\Gamma(s)}{(2\pi)^s} \sin((\pi(1-s)/2).$$

(This can be derived from the usual functional equation using the reflection formula  $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ . For  $s$  close to 1 we find

$$(s-1)\zeta(s)\Gamma(s) = 1 + O((s-1)^2),$$

and

$$\sin(\pi(1-s)/2) = -\frac{\pi}{2}(s-1) + O((s-1)^3).$$

Hence

$$\zeta(1-s) = -\frac{\pi}{(2\pi)^s} + O((s-1)^2) = -\frac{1}{2} - \frac{1}{2}\log(2\pi)(s-1) + O((s-1)^2).$$

This gives  $\zeta'(0) = -\frac{\log 2\pi}{2}$ . We now insert this into (5.2). For  $L(s) = \zeta(s)$ , this reads as

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{\log \pi}{2} + \frac{1}{2} \frac{\Gamma'(s/2)}{\Gamma(s/2)} - b + \frac{1}{s} + \frac{1}{s-1} - \sum_{\rho \neq 0,1} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right).$$

Taking the logarithmic derivative of the recurrence relation  $\Gamma(s+1) = s\Gamma(s)$  reveals

$$\frac{\Gamma'(s+1)}{\Gamma(s+1)} = \frac{1}{s} + \frac{\Gamma'(s)}{\Gamma(s)}.$$

We can use this to simplify our equation, leaving us with

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{\log \pi}{2} + \frac{1}{2} \frac{\Gamma'(s/2+1)}{\Gamma(s/2+1)} - b + \frac{1}{s-1} - \sum_{\rho \neq 0,1} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right).$$

When inserting  $s = 0$ , this sum vanishes, and we find

$$-\frac{\zeta'(0)}{\zeta(0)} = -\frac{\log \pi}{2} + \frac{1}{2} \frac{\Gamma'(1)}{\Gamma(1)} - b - 1.$$

This proves the claim, as we know  $\zeta(0) = -\frac{1}{2}$ ,  $\Gamma(1) = 1$  and the values for  $\zeta'$  and  $\Gamma'$  from above.

**Remark.** The formula  $\Gamma'(1) = -\gamma$  can be derived from the Weierstraß product

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{j=1}^{\infty} \left(1 + \frac{z}{j}\right) e^{-z/j}$$

by taking logarithmic derivatives on both sides and inserting  $z = 1$ .

**b)** This statement is false, but of course, we are supposed to show  $|\operatorname{Im}(\rho)| \geq 6$  for all the *non-trivial* roots of the zeta function. By (5.3a), we have that

$$-b = -\operatorname{Re} b = \sum_{\rho} \operatorname{Re}\left(\frac{1}{\rho}\right).$$

The idea is that if  $\operatorname{Im} \rho_1$  was small, then this sum would be so large that this equality cannot hold (remember that  $b = -0.023$  is quite small). Let  $\rho = \sigma + it$  be a root with smallest possible imaginary value. Note that with  $\rho$ , we also have roots  $1 - \sigma \pm it$  and  $\sigma - it$ , so we may assume that  $\sigma \geq \frac{1}{2}$  and that  $t > 0$ . As all contributions in the sum of (5.3a) are positive, we find

$$-b \geq \frac{1}{\sigma + it} + \frac{1}{\sigma - it} = \frac{2\sigma}{\sigma^2 + t^2} \geq \frac{1}{1 + t^2}.$$

This shows  $t \geq \sqrt{-b^{-1} - 1} \approx 6.5036$ .

### Problem 3

**a-5p)** The "only" thing left to made precise is the contour shift. The main ingrediants are (5.3b) and (5.3c).

Just for convenience, let's summarize the bounds we need for  $\frac{\zeta'}{\zeta}$ :

$$\frac{\zeta'}{\zeta}(s) = \begin{cases} O(1) & \operatorname{Re} s \geq 2, \\ O(\log |s|) & \operatorname{Re} s \leq -\frac{1}{2}, |s + 2m| \geq \frac{1}{4} \text{ for all } m \in \mathbb{N}, \\ \sum_{|\rho-s| \leq 1} \frac{1}{s-\rho} + O(1 + \log |s|), & -1 \leq \operatorname{Re} s \leq 3. \end{cases}$$

The first bound follows from  $\frac{\zeta'}{\zeta}(s) = \sum_n \Lambda(n)n^{-s}$ , the second part follows from the first bound, the functional equation and Stirling's formula. The third part is (5.3c).

By (5.3b), there are approximately  $\log T$  roots of the zeta-function with imaginary part close to  $T$  (i.e.,  $|T - \operatorname{Im} \rho| \leq 1$ ). Hence given some  $n \in \mathbb{Z}$ , the pigeonhole principle assures that it is possible to find some  $T = T_n$  with  $|n - T| \leq 1$  and  $\min_{\rho} |T - \operatorname{Im} \rho| \leq \frac{1}{\log |n|}$ . Together with (5.3c), this gives that  $\frac{\zeta'}{\zeta}(\sigma + iT) \ll (\log |T|)^2$  on that line.

The plan is now to choose some large  $n$ , and shifting the truncated integral

$$-\frac{1}{2\pi i} \int_{2-in}^{2+in} \frac{\zeta'}{\zeta}(s) \widehat{\omega}(s) ds \tag{1}$$

to  $\operatorname{Re} s = -1/2$ . This leaves us with the exercise to bound the horizontal integrals along the segments  $[2 \pm in, -1/2 \pm in]$ . By the rapid decay of  $\widehat{\omega}$  and the bounds for  $\zeta'/\zeta$ , changing the boundaries of the integral in (1) from  $[2 - in, 2 + in]$  to  $[2 - iT_n, 2 + iT_n]$  comes only with a small

cost of  $o(1)$ . So we may also assume that uniformly  $\zeta(\sigma + iT_n) \ll (\log n)^2$  along the horizontal segments. This justifies the first contour shift, and we obtain

$$\begin{aligned} \sum_n \Lambda(n) \omega(n) &= \frac{1}{2\pi i} \int_{2-in}^{2+in} \frac{\zeta'(s)}{\zeta(s)} \hat{\omega}(s) ds + o(1) \\ &= \sum_{|\operatorname{Im} \rho| \leq T_n} \hat{\omega}(\rho) + \frac{1}{2\pi i} \int_{-1/2-iT_n}^{-1/2+iT_n} \frac{\zeta'(s)}{\zeta(s)} \hat{\omega}(s) ds + o(1). \end{aligned}$$

We may let  $n \rightarrow \infty$ , obtaining

$$\sum_n \Lambda(n) \omega(n) - \sum_{\rho} \hat{\omega}(\rho) = \int_{(-1/2)} \frac{\zeta'(s)}{\zeta(s)} \hat{\omega}(s) ds.$$

In the proof of (4.4c) we can abuse the fact that  $\operatorname{Supp}(\omega) \subset [2, \infty)$  to show that  $\hat{\omega}(s) \ll \frac{2^{-s}}{|\operatorname{Im}(s)|^N}$  for all  $N$ . This shows the bound

$$\int_{(-A+1/2)} \frac{\zeta'(s)}{\zeta(s)} \hat{\omega}(s) ds \ll 2^{-A},$$

justifying the shift  $\operatorname{Re} s \rightarrow -\infty$ .

**b-5p)** The observation is that this function has large negative peaks at the primes, and when  $n = p^k$  is a prime power (to be fair, without knowing the explicit formula, this would be hard to guess). Although the solution will (implicitly) assume the Riemann conjecture, this illustrates the fact that  $\zeta$  *knows everything about the primes*.

Okay, let's analyze what's happening here. First, we have

$$\cos(\gamma_j \log x) = \operatorname{Re}(e^{i\gamma_j \log x}) = \operatorname{Re}(x^{i\gamma_j}),$$

so we are plotting the real part of the sum of  $x^{i\gamma}$  over the first few zeroes. If we want to interpret this as a sum  $\sum_{\rho} \hat{\omega}(\rho)$ , we would like to choose  $\omega$  in way such that  $\hat{\omega}(1/2 + i\gamma) \approx x^{i\gamma}$  for the zeroes we want to consider, and  $\hat{\omega}$  decaying rapidly after that range (ignoring the contribution of the trivial zeroes).

Apparently (not clear to me how to come up with this but hey it works) a convenient seems to be

$$\hat{\omega}(1/2 + i\gamma) = x^{i\gamma} \exp(-(\gamma/S)^2) = x^{s-1/2} \exp\left(\left(\frac{s-1/2}{S}\right)^2\right),$$

as this vanishes quickly once  $\gamma > S$ . We choose  $S$  to be a parameter roughly of the size of the largest zero we want to consider, which in our case is  $\gamma_{30} \approx 100$ . That's why we choose  $S = 100$ . We will later show that the weight

$$\omega(y) = \omega_{S,x}(y) = \frac{S}{2(\pi y)^{1/2}} \exp\left(-\left(\frac{S}{2} \log\left(\frac{y}{x}\right)\right)^2\right)$$

is the inverse Mellin-transform of  $\hat{\omega}$ . Now this is large if the part in the exponential vanishes, i.e., if  $x \approx y$ . On the other side, if  $y$  is not close to  $x$  then  $\log(y/x)$  becomes large (say of size  $\approx \frac{10}{S}$ ), then the factor  $\exp(-S^2/4 \log(y/x)^2)$  makes  $\omega$  decay quickly. So at least for  $x$  not too large (for large  $x$  we need a further distance between  $y$  and  $x$  to make  $\log(y/x)$  become large),  $\omega$  essentially looks like a peak at  $y = x$ . We are now ready to explain what's going on. With

the explicit formula (which we are technically not even allowed to use as  $\omega$  is not compactly supported, but whatever), we find

$$\sum_n \Lambda(n) \omega_{S,x}(n) \approx - \sum_{|\operatorname{Im} \rho| \leq S} \widehat{\omega}(\rho) \approx \sum_{j=1}^S \cos(\gamma_j \log x).$$

If now  $x \approx p^k$  is close to a prime power, the LHS is  $\approx \Lambda(n) \frac{S}{y^{1/2}}$ , large. If not, there is no term on the LHS that contributes much, so we would expect the RHS to be small.

**Prove that " $\widehat{\omega} = \omega$ ".** We put  $\widehat{\omega}$  in the inverse mellin transform to find

$$\omega(y) = \frac{1}{2\pi i} \int_{(c)} x^{s-1/2} \exp\left(\left(\frac{s-1/2}{S}\right)^2\right) y^{-s} ds$$

for all real numbers  $c$ . We substitute  $u = \frac{s-1/2}{S}$  and find

$$\omega(y) = \frac{S}{y^{1/2}} \cdot \frac{1}{2\pi i} \int_{(c)} \exp(u^2) \left(\frac{y}{x}\right)^{-Su} du.$$

Abbreviating  $v = S \log \frac{y}{x}$  shows further that

$$\frac{S}{y^{1/2}} \frac{1}{2\pi i} \int_{(c)} \exp(u^2) \exp(-uv) du = \frac{S \exp(-v^2/4)}{y^{1/2}} \cdot \frac{1}{2\pi i} \int_{(c)} \exp((u-v/2)^2) du.$$

This integral does not depend on  $c$ , hence we may wlog assume  $c = v/2$ , which reveals that this integral equals

$$\frac{1}{2\pi i} \int_{(0)} \exp(u^2) du = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-t^2) dt = \frac{1}{2\sqrt{\pi}},$$

just what we wanted.

## Problem 4

First, an aside on the weird-looking error term  $\psi(x) - x \ll xe^{-c\sqrt{\log x}}$ . On the one side it is better than every error term of the form  $x/(\log x)^A$  (for  $A \in \mathbb{R}_{>0}$  large), on the other side it is worse than every error term of the form  $x^{1-\delta}$  would be (for  $\delta \in \mathbb{R}_{>0}$  small).

Our version of the prime number theorem reads

$$\psi(x) = \sum_{p^n \leq x} \log p = x + O(xe^{-c\sqrt{\log x}})$$

for some constant  $c > 0$ . We deduce a formula for  $\pi$  in two steps. First we show that  $\psi(x)$  does not differ too much from the weighted prime-counting function

$$\psi_0(x) := \sum_{p \leq x} \log p.$$

Then we use  $\psi_0$  for partial summation, utilizing that

$$\pi(x) = \sum_{p \leq x} \frac{\log p}{\log p} = \frac{\psi_0(x)}{\log x} + \int_2^x \frac{\psi_0(t)}{t(\log t)^2} dt. \quad (2)$$

Evaluating this should be possible using the approximation for  $\psi_0(x)$ .

Let's carry this through, beginning with the estimate for  $|\psi(x) - \psi_0(x)|$ . We find

$$\psi(x) - \psi_0(x) = \sum_{p^k \leq x, k \geq 2} \log p \leq \left( \sum_{p \leq \sqrt{x}} + \sum_{p \leq x^{1/3}} + \cdots + \right) \log x$$

Note that there are at most  $\log_2 x$  summation signs which don't run over an empty set, and every index set contains (trivially) less than  $\sqrt{x}$  primes. We obtain

$$\psi(x) - \psi_0(x) \leq (\log_2 x) \sqrt{x} (\log x) \ll x^{1/2+\varepsilon}.$$

Now  $\psi_0$  satisfies the same approximation as  $\psi$ , as

$$\psi_0(x) = \psi(x) + O(x^{1/2+\varepsilon}) = x + O(xe^{-c\sqrt{\log x}}).$$

Inserting this in (1) yields

$$\pi(x) = \frac{x}{\log x} + \int_2^x \frac{1}{(\log t)^2} dt + O(xe^{-c\sqrt{\log x}}),$$

where we used that  $\int_2^x \frac{1}{t(\log t)^2} dt \ll 1$ . As

$$\int_2^x \frac{1}{(\log t)^2} dt = \left[ \text{Li}(t) - \frac{t}{\log t} \right]_2^x = \text{Li}(x) - \frac{x}{\log x} + O(1),$$

the claim follows.