Solutions to Sheet 11.

Problem 1

Let's first think about why this should be true. For $2 \neq p$ we have $g(p) = \frac{2}{p-2} \geq \frac{2}{p} = \frac{\tau(p)}{p}$. Hence for square-free numbers n we have $g(n) \geq \frac{\tau(n)}{n}$. It is easy to see that $\sum_{n \leq Q} \frac{\tau(n)}{n} \gg (\log Q)^2$ (approximate the LHS with $\left(\sum_{n \leq Q} \frac{\tau(n)}{n}\right)^2$). Hence we expect a similar lower bound (with a different constant) here. However, to make this precise we would have to show that the divisor function does not interact with the square-freeness condition too badly.

Proof using Perron's Formula. Let $G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ be the Dirichlet series attached to g. As g behaves similar to $\frac{\tau(n)}{n}$, we would hope to be able to relate g to $\zeta(s+1)^2$, which is the Dirichlet series attached to the coefficients $\frac{\tau(n)}{n}$. We write $G(s) = \zeta(s+1)^2 H(s)$, where we find in $\operatorname{Re} s > 0$

$$H(s) = \left(1 + \frac{1}{2^s}\right)\left(1 - \frac{1}{2^{s+1}}\right)^2 \prod_{p>2} \left(1 + \frac{2}{(p-2)p^s}\right)\left(1 - \frac{1}{p^{s+1}}\right)^2.$$

Factoring this out, we find that the euler factor at p is of size $1 + O(p^{-(s+2)})$, hence the euler product is absolutely (and locally uniformly) convergent whenever $\operatorname{Re} s > -1$, so H is a holomorphic function in that region and thereby does not interfere with the analysis when doing perron's formula. Also note that now G can be continued to $\operatorname{Re} s > -1$.

Now, we do what we always do. Let $T = x^{\alpha}$ (for some $\alpha \in (0,1)$) and $c = \frac{1}{\log x}$. We find by Perron's Formula

$$\sum_{n \leq x} g(n) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}T}^{c+\mathrm{i}T} G(s) x^s \frac{\mathrm{d}s}{s} + O\left(\frac{x^c}{T} \sum_{n \in \mathbb{N}} \frac{g(n)}{n^c} + \max_{n \sim x} g(n) \left(\frac{x \log x}{T}\right)\right).$$

We first inspect the O-term. As the series defining $G(s) = \zeta(s+1)^2 H(s)$ converges absolutely in $\operatorname{Re} s > 0$, we find that $g(n) \ll n^{-1+\varepsilon}$. As the pole of G at 0 has order 2, we have $\sum_{n \in \mathbb{N}} \frac{g(n)}{n^c} \ll (\log x)^2$. In particular, we find that the O-term is bounded by $O\left(\frac{x^{\varepsilon}}{T}\right)$.

We now want to shift the contour to the left, to $\operatorname{Re} s = -\frac{1}{8}$, say. We pick up a residue at $s = 0$. To compute the residue we develope everything into taylor series and find the residue to be of size $\frac{1}{2} H(0)(\log x)^2 + O(\log x)$. We obtain

$$\sum_{n \leq x} g(n) = \frac{1}{2} H(0)(\log x)^2 + \operatorname{Ver}(x,T) + \operatorname{Hor}(x,T) + O(\log x) + O\left(\frac{x^{\varepsilon}}{T}\right),$$

where $\operatorname{Ver}(x,T)$ denotes the integral along the vertical paths

$$\operatorname{Ver}(x,T) = \frac{1}{2\pi \mathrm{i}} \left(\int_{c-\mathrm{i}T}^{-1/8-\mathrm{i}T} + \int_{-1/8+\mathrm{i}T}^{c+\mathrm{i}T}\right) G(s) x^s \frac{\mathrm{d}s}{s}$$

and $\operatorname{Hor}(x,T)$ denotes the integral along the horizontal path

$$\operatorname{Hor}(x,T) = \frac{1}{2\pi \mathrm{i}} \int_{-1/8-\mathrm{i}T}^{-1/8+\mathrm{i}T} G(s) x^s \frac{\mathrm{d}s}{s}.$$

As $H(s)$ is absolutely bounded in $\operatorname{Re} s \geq -1/2$, so we can replace $G(s)$ by $\zeta(s+1)^2$ in all upcoming considerations. On the vertical lines, we have $x^s \ll x^{-1/8}$ and $\zeta(s+1)^2 \ll T^{1/4}$ (by the convexity bound), so that we find

$$\operatorname{Ver}(x,T) \ll TT^{1/4}x^{-1/8} = T^{5/4}x^{-1/8}.$$

(We could have also made use of the moment bounds, and improved this bound a lot by cutting the integral in dyadic pieces, but no need for that). For the horizontal integrals we use the convexity bound to find that

$$\frac{1}{s} \ll \frac{1}{T} \quad \text{and} \quad \zeta(s+1)^2 \ll T^{1/2} \quad \text{and} \quad x^s \ll 1,$$

revealing $\operatorname{Hor}(x,T) = O(1)$. If we choose $T = x^{\frac{1}{10}}$, we also find $\operatorname{Ver}(x,T) \ll 1$. Finally, note that $H(0) > 0$ (essentially by absolute convergence and the fact that no factor equals 0), so that

$$\sum_{n \leq x} g(n) = \frac{1}{2}H(0)(\log x)^2 + O(\log x) \gg (\log x)^2.$$

Elementary proof. Might be added later. See pages 179-181 in Brüdern's book.

Problem 2

a) If we consider C as a linear operator $\mathbb{C}^N \to \mathbb{C}^R$ and equip these spaces with the L^2-norm, the statement of the exercise is equivalent to the statement that the operator norm C and it's dual C^* coincide. This is a classical statement of functional analysis, and true in general for Hilbert spaces.

But just for the sake of completeness, here is a proof. It suffices to show that (A) implies (B), by symmetry. Assuming (A), we have

$$\text{LHS} = \sum_n \left| \sum_r c_{nr} y_r \right|^2 = \sum_{n,r,s} c_{nr}\overline{c_{ns}} y_r \overline{y_s} = \sum_r y_r \sum_n c_{nr} \sum_s \overline{c_{ns} y_s}.$$

Now we apply Cauchy-Schwartz to the sum over r, finding that

$$\text{LHS}^2 \leq \sum_r |y_r|^2 \sum_r \left| \sum_n c_{nr} \sum_s \overline{c_{ns} y_s} \right|^2 \leq \sum_r |y_r|^2 \, D \cdot \sum_n \left| \sum_s c_{ns} y_s \right|^2 = D \cdot \sum_r |y_r|^2 \cdot \text{LHS}.$$

b) We have to show that

$$\sum_r |S(\alpha_r)|^2 \ll (N + \delta^{-1}) \sum_n |a_n|^2$$

where $S(\alpha) = \sum_{M<n<M+N} a_n e(\alpha n)$ and the values α_r with pairwise distance at least δ. As in the proof from the lecture, we may shift by K without changing the absolute value of $S(\alpha)$, and may therefore assume $M \ll N$ (M might be negative). Of course we now want to apply part a), which leaves us with the task of showing that

$$\sum_{|n| \leq N} \left| \sum_r b_r e(n\alpha_r) \right|^2 \ll (N + \delta^{-1}) \sum_r |b_r|^2.$$

Because opening the absolute values and estimating the inner sums turns out to be hard, we consider a smoothed version:

$$\sum_n f(n/N) \left| \sum_r b_r e(n\alpha_r) \right|^2 \ll (N + \delta^{-1}) \sum_r |b_r|^2,$$

where f is a non-negative function with $f|_{[0,1]} = 1$ and $f(x) = 0$ for $|x| > 2$. This clearly implies the bound above.

Problem 3

We open the square and interchange sums, obtaining

$$\sum_n f(n/N)\left|\sum_r b_r e(n\alpha_r)\right|^2 = \sum_{r,s} b_r\overline{b_s}\sum_n f(n/N)e(n(\alpha_r-\alpha_s)).$$

We use the elementary inequality $|ab| \leq a^2+b^2$ to obtain

$$\begin{aligned}\cdots \ll \sum_r\sum_s(|b_r|^2+|b_s|^2)&\left|\sum_n f(n/N)e(n(\alpha_r-\alpha_s))\right| \\ &= \frac{1}{2}\sum_r |b_r|^2\sum_s\left|\sum_n f(n/N)e(n(\alpha_r-\alpha_s))\right|,\end{aligned}$$

where in the latter inequality we used many symmetrys in this sum. We now try evaluating this. First, we consider the diagonal terms with $r = s$. Here we have $\alpha_r = \alpha_s$, and we easily find that this part of the sum is bounded by $\ll N\sum_r |b_r|^2$. For the remaining part, it suffices to show that

$$\sum_{s\neq r}\left|\sum_n f(n/N)e(n(\alpha_r-\alpha_s))\right| \ll \frac{1}{\delta}.$$

The idea is that $\alpha_r - \alpha_s$ isn't too small, so we hope that there is cancellation in the sum. This is where Poisson's summation formula enters the stage. As f is Schwartz class function, its fourier transform is too and we find $\widehat{f}(y) \ll \frac{1}{1+y^2}$. Hence we obtain

$$\sum_n f(n/N)e(n(\alpha_r-\alpha_s)) = \sum_n \widehat{f}(N(\alpha_r-\alpha_s+n)) \ll N\sum_n \frac{1}{1+N^2(\alpha_r-\alpha_s+n)^2}.$$

This is easily seen to be of size $\frac{N}{1+\|\alpha_r-\alpha_s\|^2N^2}$. We are left to show that for fixed r,

$$\sum_{s\neq r}\frac{1}{1+\|\alpha_r-\alpha_s\|^2N^2} \ll \frac{1}{N\delta}.$$

It would not be good enough to just use that $\|\alpha_r-\alpha_s\| \gg \frac{1}{\delta}$. The only thing we can do to avoid this bound is to use that there are at most 2 values for s for which this is smaller than δ, at most four for which it is smaller than 2δ, etc. Hence we can bound the LHS as

$$\ll \sum_{n=1}^{\infty}\frac{1}{1+n^2\delta^2N^2},$$

which leaves us with the task of showing that

$$\sum_{n=1}^{\infty}\frac{1}{1+n^2x^2} \ll \frac{1}{x}$$

whenever $x > 0$. This is one line:

$$\text{LHS} \ll \sum_{n=1}^{\infty}\min(1,\frac{1}{n^2x^2}) \ll \sum_{n\leq 1/x} 1 + \sum_{n\geq 1/x}\frac{1}{n^2x^2} \ll \frac{1}{x}+\frac{1}{x^2}\sum_{n>1/x}\frac{1}{n^2} \ll \frac{1}{x}.$$

Problem 4

The plan is to reduce this to (6.4). We can shift indices to assume that $l = 0$. Then we are summing over multiples of k in an interval of length N. This is the same as summing over integers in an interval of length N/k. The only thing that might be in our way is the exponential term, where we have the term $e(\frac{akd}{q})$, but we would like to have $e(\frac{ad}{q})$. But as we have $(k, q) = 1$, summing over a mod q is the same as summing over ka mod q. We arrive at something which really looks like (6.4), but with an additional coprimality condition. As all terms in the sum of (6.4) are positive, the inequality still stays valid, and we are done.

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