Solutions to Sheet 12.

Problem 1

Again, we sketch a proof using Perron. We need to find out what the dirichlet function attached to g looks like. We have

$$G(s) = \sum_{n\in\mathbb{N}} g(n)n^{-s} = \prod_p \left(1 + \frac{2}{p^{s+1}}\right)$$

Remember that the Dirichlet series attached to $\frac{\tau(q)}{q}$ is given by

$$\zeta(s+1)^2 = \prod_p \left(1 + \frac{2}{p^{s+1}} + \frac{3}{p^{2(s+1)}} + \dots\right).$$

On each euler factor, the first two terms of $G(s)$ and $\zeta^2(s+1)$ coincide! So we might hope that there is a way to compare the two Dirichlet series. Indeed, writing

$$G(s) = \zeta(s+1)^2 H(s),$$

we find that $H(s)$ is given by an euler product with factor at p given by

$$\left(1 + \frac{2}{p^{s+1}}\right)\left(1 - \frac{1}{p^{s+1}}\right)^2 = 1 - \frac{3}{p^{2(s+1)}} + \frac{2}{p^{3(s+1)}} = 1 + O(p^{-2(s+1)}).$$

Now $H(s)$ is absolutely convergent and uniformly bounded in $\operatorname{Re} s \geq -\frac{1}{2} + \delta$, and we can copy the proof from exercise 1 on sheet 11.

Problem 2

Partial summation! Write

$$f(q) = \frac{1}{\varphi(q)}\left(\sum_{\chi(q)}^{*}\left|\sum_n \omega(n)\chi(n)\right|^2\right)\left(\sum_n |\omega(n)|^2\right)^{-1}.$$

Now (6.5) reads

$$\sum_{q\leq Q} qf(q) \ll N + Q^2.$$

By partial summation, we then find

$$\begin{aligned}
\sum_{R<q\leq Q} f(q) &\ll \frac{\sum_{R<q\leq Q} qf(q)}{Q} + \int_R^Q \frac{\sum_{R<q\leq t} qf(q)}{t^2}\,\mathrm{d}t \\
&\ll \frac{N+Q^2}{Q} + \int_R^Q \frac{N+t^2}{t^2}\,\mathrm{d}t \\
&\ll \frac{N}{Q} + Q + N\int_R^\infty t^{-2}dct \\
&\ll \frac{N}{R} + Q
\end{aligned}$$

and the claim follows.

Problem 3&4

a) There are (at least) two ways to set up the sifting problem. Either we sieve for those integers $n \le x^{1/2}$ such that n^2+1 is not divisible by prime numbers in some range, or we sieve for those integers $n \le x$ such that $n+1$ is a prime and n is a quadratic residue mod p for prime numbers in some different range. Let us think about the first idea, as this probably is what the exercise intents us to do. The other approach would probably be a good exercise though! We set

- $\mathcal{N} = \{x^{1/4} < n \le \sqrt{x}\}$ (this is the set of numbers we want to put into the sieve) (There is no particular reason to exclude numbers $\le x^{1/4}$, but there is also no reason to sieve for more, as we will soon see).
- $\mathcal{P} = \{2 \ne p \le x^{1/4}\}$ (this is the set of primes we want to sieve with) (this could have been chosen larger, but we will see why this is optimal (in some sense) soon).
- $\Omega_p = \{\text{Solutions to } a^2+1 \equiv 0 \bmod p\}$ (for each prime p, this is the set of residue classes mod p we want to throw out).

With this definition, we find that

$$\mathcal{N}^* = \{n \in (x^{1/4}, x^{1/2}] \mid \forall 2 \ne p \le x^{1/4} : p \nmid (n^2+1)\} \supset \{n \in (x^{1/4}, x^{1/2}] : n^2+1 \text{ prime}\}.$$

Hence, upper bounds for $\mathcal{N}^*$ deliver upper bounds for the number of primes of the form $p = n^2+1$ in the range $\sqrt{x} \le p \le x$. As there are $\ll \frac{\sqrt{x}}{\log x}$ primes up to $\sqrt{x}$, we further have

$$\#\{\text{primes of the form } p = n^2+1\} \ll \#\mathcal{N}^* + O\left(\frac{\sqrt{x}}{\log x}\right),$$

which shows that we need to show $\#\mathcal{N}^* \ll \frac{\sqrt{x}}{\log x}$ to finish the proof.

b) Note that $\rho(p) = \#\Omega_p$. We have $\rho(2) = 1$. Mod $p \ne 2$, there are 2 solutions to $\xi^2 \equiv -1$ mod p if $\left(\frac{-1}{p}\right) = 1$, i.e., if $p \equiv 1 \bmod 4$, and 0 otherwise. If $m = rs$ with $(r,s) = 1$ and we have $\rho(r)$ solutions $\xi_1^2 \equiv \cdots \equiv \xi_{\rho(r)}^2 \equiv -1 \bmod r$ and $\rho(s)$ solutions $\zeta_1^2 \equiv \cdots \equiv \zeta_{\rho(s)}^2 \equiv -1 \bmod s$, then by the chinese remainder theorem (and the fact that $a \equiv -1 \bmod m$ iff $a \equiv -1 \bmod r$ and $a \equiv -1 \bmod s$) we find that there are $\rho(r)\rho(s)$ solutions mod rs. This shows $\rho(rs) = \rho(r)\rho(s)$, as desired.

c) Using (6.9), we can bound the number of elements in $\mathcal{N}^*$. We put

$$g(q) := \mu(q)^2 \prod_{p|q} \frac{\omega(p)}{p - \omega(p)},$$

where $\omega(p) = \rho(p)$ if $p \in \mathcal{P}$ and 0 otherwise. In particular, g vanishes on even numbers. Now we have for any $Q > 1$

$$\#\mathcal{N}^* \ll (N + Q^2)\left(\sum_{q \le Q} g(q)\right)^{-1},$$

where $N = \#\mathcal{N} \ll \sqrt{x}$. The task is to find a lower bound for

$$\sum_{q \le Q} g(q),$$

where (in order not to disturb the main term) we will choose $Q \leq x^{1/4}$ (this is also why it suffices to only consider $p \leq x^{1/4}$: these are the only prime divisors that occur as divisors of numbers $\leq Q$). As ρ (and hence ω too) is multiplicative, we find that whenever q only has prime divisors $\equiv 1 \bmod 4$ that are in $\mathcal{P}$,

$$g(q) = \prod_{p|q} \frac{2}{p-2} \geq \prod_{p|q} \frac{2}{p}.$$

As $g(q)$ is supported on numbers that only have prime divisors $p \in \mathcal{P}$ with $\equiv 1 \bmod 4$, we find that for such numbers we have $g(q) \geq l(q)$, where $l(q)$ is implicitely defined via

$$P(s) = \prod_{p \equiv 1 (\mathrm{mod}\ 4)} (1 + 2p^{-s-1}) = \sum_{n \in \mathbb{N}} l(n) n^{-s}.$$

If we assume to know that $\sum_{q \leq Q} l(q) \gg \log Q$, we find for $Q \leq x^{1/4}$ that

$$\sum_{q \leq Q} g(q) \gg \sum_{q \leq Q} l(q) \gg \log Q$$

and hence

$$\#\mathcal{N}^* \ll (\sqrt{x}) \left(\sum_{q \leq x^{1/4}} g(q) \right)^{-1} \ll \frac{\sqrt{x}}{\log x},$$

which is what we wanted to show. (I don't think we need this, however).

d) It remains to show that indeed $\sum_{q \leq Q} l(q) \gg \log Q$. Our tool of choice will be Perron's formula again, or some variant thereof with smooth weights (elementary proofs are certainly possible, but probably less constructive). Let's choose c, T and write down what (4.7) says:

$$\sum_{q \leq Q} l(q) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}T}^{c+\mathrm{i}T} P(s) Q^s \frac{\mathrm{d}s}{s} + O\left(\frac{Q^c}{T} \sum_n \frac{|l(q)|}{n^c} + \max_{q \sim Q} |l(q)| \left(1 + \frac{Q \log Q}{T} \right) \right). \quad (1)$$

One of the main tasks is now to express $P(s)$ in a way that makes it possible to calculate its analytic behaviour. The hint tells us that perhaps

$$P(s) \approx L(\chi_{-4}, s+1)\zeta(s+1),$$

which is nice because we know how to deal with $\zeta(s+1)$ and $L(\chi_{-4}, s+1)$. Indeed, the Euler factor at $p \equiv 1 \pmod 4$ of $L(s+1, \chi_{-4})\zeta(s+1)$ is given by

$$(1 + p^{-(s+1)} + p^{-2(s+1)} + \cdots)^2 = 1 + 2p^{-s-1} + O(p^{-2(s+1)})$$

and at $p \equiv 3 \bmod 4$ we find

$$(1 + p^{-(s+1)} + p^{-2(s+1)} + \dots)(1 - p^{-(s+1)} + p^{-2(s+1)}) = 1 + O(p^{-2(s+1)}).$$

We can use the power series expansion

$$(1+x)^{-1} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

to deduce that the euler factors of $H(s+1) = P(s)(\zeta(s+1)L(\chi_{-4}, s+1)^{-1}$ all lie in $1 + O(p^{-2(s+1)})$, which gives that $H(s)$ is holomorphic and absolutely convergent (hence uniformly bounded) in $\mathrm{Re}\, s \geq 2/3$.

e) First, we will work through how one can find the bound using (1). Then we will discuss how one could have used a smooth weight to simplify the analysis.

Okay, so let's start with (1). As usual, we choose $c = 1/\log Q$ and $T = Q^\alpha$ for some $\alpha \in (0,1)$. We first inspect the O-term. As $P(s)$ is absolutely convergent in $\operatorname{Re} s > 0$, we find that $l(q) \ll q^{\varepsilon - 1}$. As $L(\chi_{-4}, 1) \neq 0$, $P(s)$ has at most a simple pole at 0, hence we find $\sum_n g(n) n^{-c} \ll Q^\varepsilon$, and the whole O-term is bounded by $O(Q^\varepsilon/T)$. By the product expansion and the analytic continuations of ζ and L, we can continue P to a meromorphic function in $\operatorname{Re} s > -1/3$, and we know that the only pole is at $s = 0$ with residue $H(1)L(\chi_{-4}, 1) \neq 0$. We find that

$$\operatorname{Res}_{s=0}\left(P(s)\frac{Q^s}{s}\right) = H(1)L(\chi_{-4}, 1)(\log Q) + C$$

for some constant C independent of Q. Now we have to shift the contour, and every contour a tad to the left of $\operatorname{Re} s = 0$ suffices. Hence we might choose $\operatorname{Re} s = -1/8$. The remaining integral along the path $\gamma_1 \cup \gamma_2 \cup \gamma_3$ where

$$\gamma_1 = [c - \mathrm{i}T, -1/8 - \mathrm{i}T], \quad \gamma_2 = [-1/8 - \mathrm{i}T, -1/8, \mathrm{i}T], \quad \gamma_3 = [-1/8 + \mathrm{i}T, c + \mathrm{i}T]$$

can be easily bounded using the convexity bound, which states that in this region

$$\zeta(s) \ll (1 + |s|)^{\frac{1-\sigma}{2} + \varepsilon} \quad \text{and} \quad L(s, \chi_{-4}) \ll (1 + |s|)^{\frac{1-\sigma}{2} + \varepsilon}.$$

In total, after choosing T (more precisely, α) appropiately small, no integal contributes more than $O(1)$. This shows the asymptotic

$$\sum_{q \le Q} l(q) = H(1)L(1, \chi_{-4})(\log Q) + O(1),$$

and we in particular find $\sum_{q \le Q} l(q) \gg \log Q$.

Using a smooth weight. We can make our life a lot easier if we choose some smooth weight ω with support in $[0,1]$ and $\omega|_{[0,1/2]} = 1$. With this choice, the derivative of ω is compactly supported. Note that by integration by parts and in $\operatorname{Re} s > 0$ we have

$$\widehat{\omega}(s) = \int_0^\infty \omega(x) x^{s-1} \,\mathrm{d}x = -\frac{1}{s}\int_0^\infty \omega'(x) x^s \,\mathrm{d}x = -\frac{1}{s}\mathcal{M}(\omega')(s). \tag{2}$$

Here, $\mathcal{M}(\omega')$ is holomorphic on $\mathbb{C}$ and rapidly decaying on vertical lines by (4.4). Therefore, (2) gives a meromorphic continuation of $\widehat{\omega}$ to $\mathbb{C}$ with a simple pole at 0, and we find that $\widehat{\omega}$ is also rapidly decaying on vertical lines.

We find

$$\sum_{q \le Q} l(q) \ge \sum_{q \in \mathbb{N}} l(q)\omega(q/Q) = \frac{1}{2\pi\mathrm{i}}\int_{(c)} P(s) Q^s \widehat{\omega}(s) \,\mathrm{d}s.$$

This integral is converging absolutely. Now shifting the integral to the left is easy as $\widehat{\omega}$ is eating through everything (note that ζ and L dont grow too fast by the convexity bound) and the horizontal integrals vanish in $\lim_{T\to\infty}$. We find

$$\frac{1}{2\pi\mathrm{i}}\int_{(c)} P(s)Q^s\widehat{\omega}(s)\,\mathrm{d}s = \frac{1}{2\pi\mathrm{i}}\int_{(-1/8)} P(s)Q^s\widehat{\omega}(s)\,\mathrm{d}s + \operatorname{Res}_{s=0}\left(P(s)Q^s\widehat{\omega}(s)\right).$$

As before, the residue is of size $\gg \log Q$, and the remaining integral is absolutely convergent, thereby of size $O(Q^{-1/8})$.

Max von Consbruch, email: s6mavonc@uni-bonn.de. Date: January 24, 2023