Solutions to Sheet 13.

Problem 1

I somehow couldn't make Selberg's sieve (7.5) work, so we will just use the large sieve (6.9) again. We set $F(x) = \prod_{j=1}^{k}(q_j x + r_j)$.

First we need to set up the large sieve. We make the following choices:

- $\mathcal{N} = \{n \leq x\}$
- $\mathcal{P} = \{p \leq \sqrt{x}\}$
- $\Omega_p = \{a \in \mathbb{Z}/p\mathbb{Z} \mid F(a) \equiv 0 \pmod p\}$.

With this setup, we sift for those $n \leq x$ such that all the numbers $q_j n + r_j$ have no prime divisors $\leq \sqrt{x}$, i.e.,

$$\#\mathcal{N}^* = \{n \leq x : p \mid q_j n + r_j \implies p > \sqrt{x}\}.$$

As $q_j n + r_j$ might be larger than x, this does not guarantee that indeed all remaining numbers are primes, but we still get an upper bound, so we are good. There are some subteleties with this setup. First, note that it might happen that $\omega(p) = \#\Omega_p = p$. In this case however we find that $\mathcal{N}^* = \emptyset$, and any upper bound holds. Our long-term goal is the following. As $\deg(F) = k$, we expect $\omega(p) = k$ for *most* primes p, and hence we should have something like

$$g(m) = \mu^2(m) \prod_{p|m} \frac{\omega(p)}{p - \omega(p)} \approx \mu^2(m) \prod_{p|m} \frac{k}{p} \approx \mu^2(m) \frac{\tau_k(m)}{m}.$$

From here we want to proceed as on the previous sheets (using Perron's formula) to show that

$$\sum_{m \leq x} g(m) \gg (\log x)^k.$$

And indeed, we can show that *most* means *all but finitely many.* First, we throw out all p that divide one of the q_j. Then for all remaining p, there is for every j a unique residue a_j such that $p \mid q_j a_j + r_j$. If F had a multiple zero mod p, we'd have that $a_j = a_i$ for some $j \neq a_i$, which immediately implies that

$$p \,\Big|\, \det \begin{pmatrix} q_j & q_j \\ r_i & r_j \end{pmatrix}.$$

But all those determinants are non-vanishing, so that there is only a finite number of such p. Hence we set $\mathcal{P}' = \mathcal{P} \setminus \{\text{finite set of primes}\}$, and we can apply the methods the last two sheets to find lower bounds for $\sum_{n \leq x} g(n)$. Everything goes through, as removing a finite set of primes only changes the Pirichlet function we consider in the Perron approach by some finite euler product (which thereby is holomorphic). (Although, if we write $G(s) = \zeta(s+1)^k H(s)$ with $H(s)$ absolutely convergent a bit to the left of $\operatorname{Re} s = 0$, we would still have to somehow show that $H(0) \neq 0$. But I guess this can be done elementarily).

Problem 2

I am a bit annoyed that this is in the exercises, this is purely elementary. But here we go.

Given a function f which is continuous and monotonic, we have to show that

$$|D_f(x;q,a)| = \left| \sum_{n\le x \ n\equiv a (\text{mod })q} f(n) - \frac{1}{\varphi(q)} \sum_{n\le x,\ (n,q)=1} f(n) \right| \le 2(|f(1)| + |f(x)|). \qquad (1)$$

By symmetry, we may assume that f is positive and monotonely increasing. We split up $D_f(x;q,a)$, writing $K = \lfloor \frac{x-a}{q} \rfloor - 1$:

$$D_f(x;q,a) = -\frac{1}{\varphi(q)} \sum_{n<\min(a,x),\ (n,q)=1} f(n) + \sum_{k=0}^{K-1} \left(f(kq+a) - \frac{1}{\varphi(q)} \sum_{kq+a\le n<(k+1)q+a} f(n) \right)$$
$$+ f((K+1)q+a)\delta_{x\ge a} - \frac{1}{\varphi(q)} \sum_{\lfloor \frac{x-a}{q} \rfloor q+a \le n\le x \ (n,q)=1} f(n).$$

The first and the last sum have combined size $\ge -(f(1)+f(x))$, together with the summand $f((K+1)q+a)\delta_{x\ge a}$ we find that everything except the big sum in the middle is of absolute size $\le (f(1)+f(x))$. This is easily seen by monotonicity of f and the fact that these sums run over sets of combined cardinality at most $\varphi(q)$. The sum in the middle telescopes. Indeed, by monotonicity we have

$$f(kq+a) \le \frac{1}{\varphi(q)} \sum_{kq+a\le n<(k+1)q+a} f(n) \le f((k+1)q+a),$$

and these inequalities can be combined into the bound

$$0 \ge \sum_{k=0}^{\lfloor \frac{x-a}{q} \rfloor} \left(f(kq+a) - \frac{1}{\varphi(q)} \sum_{kq+a\le n<(k+1)q+a} f(n) \right) \ge -\frac{1}{\varphi(q)} \sum_{Kq+a\le n<(K+1)q+a} f(n).$$

The RHS is $\ge -f(x)$. Combining everything, we showed

$$|D_f(x;q,a)| \le f(1) + 2f(x) \le 2(f(1)+f(x)).$$

Problem 3&4

By (5.13), we have for any $A>0$ any $q<(\log x)^A$ and any primitive χ mod q that

$$\sum_{n\le x} \Lambda(n)\chi(n) \ll_A x\exp(-c\sqrt{\log x}). \qquad (2)$$

For any $C>0$, this can be relaxed to

$$\sum_{n\le x} \Lambda(n)\chi(n) \ll_A x\exp(-c\sqrt{\log x}) \ll x/(\log x)^C. \qquad (3)$$

a) We first show that

$$\sum_{n\le x} \Lambda(n)\chi(n) \ll_{\delta,B} q^\delta x/(\log x)^B \qquad (4)$$

for all $\delta, B > 0$ and $q > 2$. First, note that this claim is a direct consequence of the prime number theorem once $q > (\log x)^{B/\delta}$. Hence it suffices to check this for $q < (\log x)^{B/\delta}$. But in this region we can apply (3) with $A = B/\delta$ and $C = B$. We obtain the inequalities

$$\sum_{n \le x} \Lambda(n)\chi(n) = \begin{cases} O_{B/\delta}(x/(\log x)^B) & q < (\log x)^{B/\delta} \\ O(x) & q \ge (\log x)^{B/\delta}. \end{cases}$$

It is easily seen that these inequalities combine into (4).

Now we have to show that

$$\sum_{p \le x} \chi(p) \ll_{\delta,B} q^{\delta} x/(\log x)^B. \tag{5}$$

This is akin to deriving the prime number theorem for π from the prime number theorem for ψ. Namely, it is another exercise in partial summation. This will just be a sketch of the arguments, look at sheet 8 for more details. First note that the higher prime powers do not contribute much, we have

$$\sum_{n \le x} \Lambda(n)\chi(n) = \sum_{p^k \le x} (\log p)\chi(p^k) = \sum_{p \le x} (\log p)\chi(p) + O(x^{1/2+\varepsilon}). \tag{6}$$

Now, partial summation. We find

$$\sum_{p \le x} \chi(p) = (\log x)^{-1} \sum_{p \le x} (\log p)\chi(p) + \int_1^x \frac{\sum_{p \le t} (\log p)\chi(p)}{t(\log t)^2} \, \mathrm{d}t.$$

Repeated use of (6) and (4) yields the claim: The integral can be bounded by

$$\ll q^{\delta} \int_1^x \frac{1}{(\log t)^{2+B}} \, \mathrm{d}t \le q^{\delta} \left(x^{1/2} + \int_{x^{1/2}}^x \frac{2^B}{(\log x)^B} \, \mathrm{d}t \right) \ll_B q^{\delta} x (\log x)^{-B},$$

the remaining term is easily seen to be of that size too.

Remark. This bound is even true for all $\chi \ne \chi_0 \bmod q$. Indeed, suppose non-principal $\chi \bmod q$ is implied by $\chi_1 \bmod q_1$, with χ_1 primitive. We can compare the sums over χ and χ_1, as

$$\sum_{n \le x} \chi(n)\Lambda(n) = \sum_{n \le x,\ (n,q)=1} \chi(n)\Lambda(n) + O(\omega(q)(\log x)^2),$$

and in the latter sum we can replace χ with χ_1. We also have

$$\sum_{n \le x} \chi_1(n)\Lambda(n) \ll_{\delta,B} q_1^{\delta} x (\log x)^{-B} \ll q^{\delta} x (\log x)^{-B},$$

and the claim follows after combining the previous two equations with (4).

b) Now we are supposed to show that

$$\sum_{n \le x} \chi(n)\mu(n) \ll_{\delta,B} q^{\delta} x/(\log x)^B. \tag{7}$$

As the hint commands, we try to make use of the bijection

$$\{\text{square-free numbers } n \le x\} \leftrightarrow \{(m,p) \mid mp \le x \wedge p_m < p \wedge m \ \square\text{-free}\}.$$

Here (and from now on), p_m denotes the largest prime divisor of m. Let's just insert this and see what we get.

$$\sum_{n \le x} \mu(n)\chi(n) = \sum_{n \le x \ \square \nmid n} \mu(n)\chi(n) = 1 - \sum_{m \le x} \chi(m)\mu(m) \sum_{p_m \le p \le x/m} \chi(p).$$

Applying (5) to the inner sum and using that that $\mu(m)\chi(m) \ll 1$ (and writing $\sum_{p_m \le p \le x}$ as $\sum_{p\le x} - \sum_{p \le p_m}$) yields

$$\text{RHS} \ll_{\delta,B} 1 + \sum_{mp_m \le x,\ \square \nmid m} q^\delta \frac{x}{m} (\log \tfrac{x}{m})^{-B}.$$

Our main task is now to bound $(\log \frac{x}{m})$. As $mp_m \le x$ and $m \le p_m^{\omega(m)}$ (note that m is squarefree), we find that $m^{\frac{\omega(m)+1}{\omega(m)}} \le x$, which implies that $x^{\frac{1}{\omega(m)+1}} \le \frac{x}{m}$. As $B > 0$, this implies that

$$\text{RHS} \ll 1 + \sum_{\square \nmid m \le x} q^\delta \frac{x}{m} (\log x)^{-B} (1 + \omega(m))^B$$

We use that $(1+\omega(m))^B \le 2^B \omega(m)^B \ll_B \omega(m)^B$. Now we only need to find a bound for the sum

$$\sum_{\square \nmid m \le x} \frac{\omega(m)^B}{m}.$$

I struggeled very hard with bounding this, at some point Bart thold me how it's done: We show that

$$\sum_{\square \nmid m \le x} \frac{\omega(m)^B}{m} \ll (\log x)(\log\log x)^B.$$

For $B = 0$, this is obvious. We now do induction on B, and basically just reorder the sum.

$$\sum_{\square \nmid m \le x} \frac{\omega(m)^B}{m} \le \sum_{m \le x} \frac{\omega(m)^{B-1}}{m} \sum_{p \mid m} 1 = \sum_{p \le x} \frac{1}{p} \sum_{m \le x/p} \frac{\omega(m)^{B-1}}{m}$$
$$\ll \sum_{p \le x} \frac{1}{p} (\log x)(\log\log x)^{B-1} \ll (\log x)(\log\log x)^B.$$

In the last step we used Merten's theorem for the sum of reciprocals of the primes. This finishes the proof of (7).

c) The first challenge is to even find out what we are supposed to show. We will show that

$$D_\mu(x; q, a) := \sum_{n \le x\ n \equiv a \bmod q} \mu(n) - \frac{1}{\varphi(q)} \sum_{n \le x} \mu(n) \ll x(\log x)^{-A}, \tag{8}$$

which is similar to (8.3) as $\|\mu\|_2 = \sqrt{\sum_{n\le x} \mu(n)^2} \approx x^{1/2}$. The function $D_\mu(x; q, a)$ measures how far μ fails to be equidestributed in the residue class a mod q up to x. Trickery with orthogonality relations quickliy reveals

$$D_\mu(x; q, a) = \frac{1}{\varphi(q)} \sum_{\chi \ne \chi_0} \overline{\chi}(a) \sum_{n \le x} \mu(n)\chi(n), \tag{9}$$

this form was already hinted at in (8.3). And this is really nice, as now we can make use of the previous parts of the exercise. If we just insert (7) in (9), we directly obtain

$$D_\mu(x; q, a) \ll_{B,\delta} q^\delta x (\log x)^{-B}.$$

This is not strong enough, as the bound in (8) should be uniform in q, and the RHS explodes if q is large. However, we aren't far from solving this exercise. Let $A > 0$ be given. In the range $q < (\log x)^{2A}$, the previous inequality gives the desired bound (choose $\delta = 1$ and $B = 3A$). If q

is larger, there are only a few numbers we sum over. Indeed, if $q > (\log x)^{2A}$, we can trivially bound from the definition (8):

$$\sum_{n\leq x \ n\equiv a \bmod q} \mu(n) - \frac{1}{\varphi(q)} \sum_{n\leq x} \mu(n) \ll 1 + \frac{x}{q} + \frac{\log\log q}{q} x \ll x(\log x)^{-A}.$$

(Remember that $\varphi(q) \gg \frac{q}{\log\log q} \gg q^{1-\varepsilon}$ for every $\varepsilon > 0$). These two cases combine into the desired bound. GGWP.

Max von Consbruch, email: s6mavonc@uni-bonn.de. Date: February 1, 2023