Prof. Dr. Valentin Blomer winter term 2022/23

Analytic Number Theory
Problem Set 1

Problem 1. Show that there are arbitrarily long intervals containing no primes in two ways:

a) Construct explicitly a number a_n such that $a_n, a_n + 1, \ldots, a_n + n$ are composite numbers.

b) Use Problem 1g) of Set 0 to show that there exist a constant $c > 0$ and infinitely many n such that $[n, n + c \log n]$ contains no prime.

Problem 2. a) Show that there are precisely two arithmetic functions α with $\alpha * \alpha = \mathbf{1}$, at most one of which is multiplicative. *Hint:* Define α recursively.
b) Considering the power series of $(1 - x)^{-1/2}$, show that one choice for α is given by $\alpha(p^n) = \frac{(2n)!}{4^n (n!)^2}$, extended multiplicatively to all integers.

Problem 3. a) Let $a, n \in \mathbb{N}$. Show that

$$\sum_{d \mid n} a^{\omega(d)} = \tau(n^a).$$

b) Show that $\sum_{d \mid n} \mu(d)\tau(d) = (-1)^{\omega(n)}$.
c) Show that the sum of the primitive n-th roots of unity equals $\mu(n)$ (in particular, it vanishes unless n is squarefree):

$$\sum_{\substack{1 \leqslant m \leqslant n \\ (m,n)=1}} e^{2\pi i m/n} = \mu(n).$$

Hint: Write the characteristic function on $(m, n) = 1$ as $\sum_{d \mid (m,n)} \mu(d)$ (why?).

Problem 4. Show that there exists $C \in \mathbb{R}$ such that

$$\sum_{n \leqslant x} n^{-1/2} = 2x^{1/2} + C + O(x^{-1/2}).$$

Due: Tuesday, Oct 18

Solutions to Sheet 1

Exercise 1

1. We may choose $a_n = (2n+2)! + 2$. Note that now $2 \mid (2n+2)! + 2$, $3 \mid (2n+2)! + 3$, etc.

2. We already know that $\pi(x) \le M\frac{x}{\log(x)}$ for some $M > 0$ and $x > 2$. We solve the exercise by assuming that for all $c > 0$ there are only finitely many $n \in \mathbb{N}$ such that the interval $[n, n + c\log(n)]$ does not contain a prime, which ultimately will result in a contradiction to the statement above.

Let us make a choice for c and count the number of primes in $[x, 2x]$, for some large number x. We trivially obtain

$$\pi(2x) - \pi(x) \le M\frac{2x}{\log(2x)}.$$

By our assumption, if x is large enough, there is no $n \in \mathbb{N} \cap [x, 2x]$ such the interval $[n, n+c\log(n)]$ does not contain a prime. Let us define numbers a_k such that $a_0 = [x]+1$, $a_{k+1} = a_k + c\log(a_k)$. Further, let $N \in \mathbb{N}$ be defined via $a_{N-1} \le 2x < a_N$. As every interval $[a_k, a_{k+1}]$ contains a prime, this yields the estimate $N \le \pi(2x) - \pi(x)$. Also, for $k < N$ we have $a_{k+1} - a_k \le c\log(2x)$. This yields the estimate

$$\frac{x}{c\log(2x)} \le N \le \pi(2x) - \pi(x) \le 2M\frac{x}{\log(2x)},$$

which is a contradiction once we choose $c < \frac{1}{2M}$.

Notes after correcting.

- Main reason for point-loss: Messy write-ups
- Common mistake: Whenever we have inequalities $a \le b$ and $c \le d$, we cannot deduce $a - c \le b - d$. For that reason, we cannot effectively bound $\pi(x+h) - \pi(x)$ for small values of h by only knowing an upper bound for π.
- $f(x) = O(g(x))$ does not imply that $\frac{f(x)}{g(x)}$ approaches some value $C \in \mathbb{R}$ as $x \to \infty$. Rather, it implies that the absolute value of this fraction is bounded.

Exercise 2

1. Via $\alpha \star \alpha = 1$, we obtain $\alpha(1) = \pm 1$. Having defined $\alpha(n)$ for values $n \le N$, $\alpha(N)$ is uniquely determined by the equation

$$1 = \sum_{d \mid N} \alpha(d)\alpha(N/d) = 2\alpha(N) + \sum_{d \mid N, d \ne 1, N} \alpha(d)\alpha(N/d).$$

Any choice of $\alpha(1)$ thereby extends to an arithmetic function with $\alpha \star \alpha = 1$, and α cannot be multiplicative if $\alpha(1) \ne 1$.

2. We set $\alpha(1) = 1$ define $\alpha(p^n)$ via the taylor series expansion of $(1-x)^{\frac{-1}{2}}$:

$$\sum_{n \in \mathbb{N}} \alpha(p^n)x^n = (1-x)^{\frac{-1}{2}}$$

(Note that $(1-x)^{\frac{-1}{2}}$ is holomorphic in some neighbourhood around 0) and extend α to a multiplicative function via $\alpha(n) = \prod_p \alpha(p^{v_p(n)})$. By the formula for multiplying taylor series, we find

$$\sum_{n\in\mathbb{N}} x^n = \frac{1}{1-x} = \left(\frac{1}{1-x}\right)^{2\frac{1}{2}} = \sum_{k\in\mathbb{N}} x^k \sum_{0\le l\le k} \alpha(p^l)\alpha(p^{k-l}).$$

After equating coefficients, this gives

$$\sum_{0\le l\le k} \alpha(p^l)\alpha(p^{k-1}) = 1,$$

i.e. $\alpha \star \alpha = 1$. (Note that α and 1 are multiplicative, so it suffices to check the equality on prime-powers). Basic analysis also reveals that α is now given by $\alpha(p^n) = \frac{(2n)!}{4^n(n!)^2}$, as demanded by the exercise.

Notes after correcting.

- Part 1 was relatively easy.
- For part 2, one can also use that $\alpha(p^n) = (-1)^n\binom{-\frac{1}{2}}{n}$ and deduce $\alpha\star\alpha = 1$ using formulas for binomial coefficients. This does not use generating functions, but it is messy.

Exercise 3

1. It is easily seen that both sides are multiplicative, and we may reduce to the case $n = p^k$, p prime. The LHS becomes $1 + ak$, the RHS becomes $1 + ak$ too, and we are done.

2. Again, both sides are multiplicative. (For the RHS, note that the product and the convolution of any two multiplicative functions is multiplicative, and that RHS $= 1 \star (\mu\tau)$.) For $n = 1$, we find LHS = RHS = 1. For prime powers $n = p^k$ with $k \ge 1$, we find

$$\text{LHS} = \mu(p^0)\tau(p^0) + \mu(p^1)\tau(p^1) + \underbrace{\mu(p^2)\tau(p^2) + \cdots + \mu(p^n)\tau(p^n)}_{=0 \text{ as } \mu(p^k)=0 \text{ for } k\ge 2.} = 1 - 2 = -1.$$

As in this case we also have RHS $= -1$, we are done.

3. We write $e(\theta)$ for $\mathrm{e}^{2\pi \mathrm{i}\theta}$. We first get rid of the condition $(m, n) = 1$ via adding the term

$$\eta((m,n)) = (1 \star \mu)((m,n))$$

to each summand, obtaining

$$\text{LHS} = \sum_{1\le m\le n \text{ and } (m,n)=1} e(m/n) \sum_{d|(m,n)} \mu(d).$$

We change the order of summation, bringing d to the outer sum, writing $m = dk$ for $d \mid n$. This gives

$$\text{LHS} = \sum_{d|n} \mu(d) \sum_{k\le n/d} e(\tfrac{k}{n/d}).$$

Now the inner sum goes over all n/d-th roots of unity, and thereby equals 0 whenever $n/d > 1$. Hence we find LHS = RHS, as desired.

Notes after correcting.

- Part 2 can be done in multiple ways, one can for example use binomial coefficient stuff to check the identity directly (for general n and not only prime-powers).
- The trick used in part 3 is quite commonly used and should be added to your Analytic number theory toolkit!

Exercise 4

We use summation by parts, setting $a_n = 1$ and $g(x) = \frac{1}{\sqrt{x}}$. We find

$$\sum_{1 \leq n \leq x} \frac{1}{\sqrt{n}} = \frac{[x]}{\sqrt{x}} + \frac{1}{2}\int_1^x \frac{[t]}{t^{\frac{3}{2}}}\,\mathrm{d}t = \sqrt{x} - \frac{\{x\}}{\sqrt{x}} + \frac{1}{2}\int_1^x \frac{1}{\sqrt{t}} - \frac{\{t\}}{t^{3/2}}\,\mathrm{d}t.$$

We have $\{x\}/\sqrt{x} = O(x^{-\frac{1}{2}})$,

$$\frac{1}{2}\int_1^x \frac{1}{\sqrt{t}}\,\mathrm{d}t = [\sqrt{t}]_1^x = \sqrt{x} - 1$$

and

$$\frac{1}{2}\int_1^x \frac{\{t\}}{t^{3/2}}\,\mathrm{d}t = \frac{1}{2}\int_1^\infty \frac{\{t\}}{t^{3/2}}\,\mathrm{d}t - \frac{1}{2}\int_x^\infty \frac{\{t\}}{t^{3/2}}\,\mathrm{d}t.$$

Here the first integral converges, and the second integral lies within $O(x^{-1/2})$. The claim follows, with

$$C = \frac{1}{2}\int_1^\infty \frac{\{t\}}{t^{3/2}}\,\mathrm{d}t - 1.$$

Notes after correcting.

- Common mistake: Errors while calculating the integral (but I am sure this will get better as the course progresses).

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Prof. Dr. Valentin Blomer winter term 2022/23

Analytic Number Theory
Problem Set 2

Problem 1. A number $n \in \mathbb{N}$ is called squarefull if every prime divisor of n occurs at least with multiplicity 2. Write down all squarefull numbers $\leqslant 100$ that are not perfect squares.

Show that every squarefull number n can be written uniquely in the form $n = a^2b^3$ with b squarefree and conclude that

$$\sum_{n \text{ squarefull}} \frac{1}{n^s} = \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)}.$$

Problem 2. Write

$$\sum_{a\in\mathbb{N}} \sum_{b\in\mathbb{N}} \frac{(a,b)}{a^s b^t}$$

in terms of the Riemann zeta function. Here (a,b) denotes the greatest common divisor. In what region $(s,t) \in \mathbb{C}\times\mathbb{C}$ does the double series converge absolutely?
Hint: Sort the double sum by the value $d = (a,b)$.

Problem 3. In $\Re s > 1$ let

$$\psi(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} \dots \quad \text{and} \quad \tilde{\psi}(s) = 1 + \frac{1}{2^s} - \frac{2}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} - \frac{2}{6^s} \dots$$

a) Show that $\psi(s) = (1-2^{1-s})\zeta(s)$ and $\tilde{\psi}(s) = (1-3^{1-s})\zeta(s)$.
b) Show that $\psi(s)$ and $\tilde{\psi}(s)$ converge (conditionally) in $\Re s > 0$ (use (1.11)).
c) Conclude from the first identity in part a) that $\zeta(s)$ can be extended meromorphically to the region $\Re s > 0$ with poles at most in the set $1 + \frac{2\pi i}{\log 2}\mathbb{Z}$. Conclude an analogous statement from the second identity.
d) Show that $\log 2/\log 3$ is irrational and conclude that ζ is holomorphic in $\Re s > 0$ with the exception of a pole at $s = 1$ (use (1.13) to deduce that this pole exists).

Problem 4. Show $\sum_{n\leqslant x} \phi(n)/n = \zeta(2)^{-1}x + O(\log x)$. *Hint:* prove and use the convolution formula $\phi(n)/n = \sum_{d|n} \mu(d)/d$.

Due: Tuesday, Oct 25

Solutions to Sheet 2

Exercise 1

1. The squarefull non-squares up to onehundred are $8, 27, 32, 72$.

2. It suffices to show that any squarefull prime power can be written uniquely as $p^k = a^2 b^3$ with b square-free. But this is the same as writing $k = 2a + 3b$ with $0 \leq b \leq 1$, and this is possible in a unique way once $k \geq 2$.

3. Using the above and that b is square-free iff $\mu^2(b) = 1$, we may write

$$\sum_{n \text{ squarefull}} n^{-s} = \sum_{a,b} \frac{\mu^2(b)}{a^{2s} b^{3s}} = \zeta(2s) \sum_b \mu^2(b) b^{-3s}.$$

We can extend the Dirichlet series of μ^2 into an Euler product, obtaining

$$\sum_{n \in \mathbb{N}} \mu^2(n) n^{-s} = \prod_p \left(1 + p^{-s}\right) = \prod_p \frac{(1 - p^{-s})^{-1}}{(1 - p^{-2s})^{-1}} = \frac{\zeta(s)}{\zeta(2s)}.$$

(In the second-to-last equality we used $(1 + x)(1 - x) = 1 - x^2$.) We find

$$\sum_b \mu^2(b) b^{-3s} = \frac{\zeta(3s)}{\zeta(6s)},$$

done.

Exercise 2

This is just a messy calculation. We somehow want to get of the (a, b)-symbol in the sum. We do so by using that given $a, b \in \mathbb{N}$, we find unique coprime numbers k, l with $a = k(a, b)$ and $b = l(a, b)$. Now summing over all possible gcds d yields

$$\sum_{a,b} \frac{(a, b)}{a^s b^t} = \sum_d \frac{d}{d^{s+t}} \sum_{k,l \in \mathbb{N} \text{ coprime}} k^{-s} l^{-t} = \zeta(s + t - 1) \sum_{k,l} k^{-s} l^{-t} \sum_{e | (k,l)} \mu(e)$$

where we rephrased the coprimality condition on k and l using the trick from the last sheet. Now we rewrite

$$\sum_{k,l} k^{-s} l^{-t} \sum_{e|(k,l)} \mu(e) = \sum_e \mu(e) \sum_{k,l} (ke)^{-s} (le)^{-t} = \frac{\zeta(s)\zeta(t)}{\zeta(s + t)},$$

obtaining

$$\sum_{a,b} \frac{(a, b)}{a^s b^t} = \frac{\zeta(s + t - 1)\zeta(s)\zeta(t)}{\zeta(s + t)}.$$

Tracing through this calculation, we find that it is sufficient for absolute convergence to have $\Re(s) > 1$ and $\Re(t) > 1$. These conditions are easily seen to be necessary too (the sub-sums with $a = 1$ or $b = 1$ diverge otherwise).

Notes after correcting.

- Even though it is easily seen that the double sum cannot converge absolutely whenever (say) $\Re(s) \leq 1$, this does immediately follow from the divergence of the series in the ζ-representation! The reason is that it is that we split the series in the first equality. It is possible to split a convergent series into divergent ones, as for example

$$\sum_{n\in\mathbb{N}} 0 = \sum_{n\in\mathbb{N}} (1-1) \neq \sum_n 1 - \sum_n 1.$$

Exercise 3

1. We have

$$\psi(s) = \sum_n n^{-s} - 2\sum_n (2n)^{-s}$$

and

$$\tilde{\psi}(s) = \sum_n n^{-s} - 3\sum_n (3n)^{-s}.$$

2. Using the Leibniz criterion, we see that the series converge conditionally on the positive real line, and thereby for $\Re s > 0$ by theorem (1.10). Alternatively, one can use (1.11) to see that the abscissa of convergence is given by

$$\sigma_0 = \limsup_{N\to\infty} \frac{\log \left|\sum_{n\leq N} (-1)^n\right|}{\log N} = 0.$$

3. As both ψ and $\tilde{\psi}$ are holomorphic in $\Re s > 0$, ζ can only have a pole whenever $(1-2^{1-s})$ and $(1-3^{1-s})$ vanish. But this is the case whenever

$$1 = 2^{1-s} = \mathrm{e}^{(\log 2)(1-s)} \quad \Leftrightarrow \quad (\log 2)(1-s) \in 2\pi\mathrm{i}\mathbb{Z}$$

and

$$1 = 3^{1-s} = \mathrm{e}^{(\log 3)(1-s)} \quad \Leftrightarrow \quad (\log 3)(1-s) \in 2\pi\mathrm{i}\mathbb{Z}.$$

4. If $\log 2/\log 3 = p/q$ was rational, we'd find that $2^q = 3^p$, contradiction. Hence the two sets $(\log 2)^{-1}(2\pi i\mathbb{Z})$ and $(\log 3)^{-1}(2\pi i\mathbb{Z})$ have intersection the set $\{0\}$. Thereby, ζ cannot have a pole away from $s = 1$. There it has a pole from a theorem in the lecture, and it is a simple pole as $(2^{1-s} - 1)$ has a simple zero at $s = 1$.

Exercise 4

We know that the d-th cyclotomic polynomial $\Phi_d(x)$ has degree $\varphi(d)$, and that $\prod_{d|n} \Phi_d(x) = x^n - 1$. Hence

$$\sum_{d|n} \varphi(d) = \sum_{d|n} \deg \Phi_d = \deg \left(\prod_{d|n} \Phi_d\right) = \deg(x^n - 1) = n,$$

hence (by Möbius-inversion)

$$\varphi(n) = (\mu \star \mathrm{id})(n) = \sum_{d|n} \frac{n}{d}\mu(d).$$

Now we find

$$\sum_{n\le x}\varphi(n)/n=\sum_{n\le x}\frac{1}{n}\sum_{d|n}\frac{n}{d}\mu(d)=\sum_{d\le x}\frac{\mu(d)}{d}\sum_{k:kd\le x}1=\sum_{d\le x}\frac{\mu(d)}{d}\left[\frac{x}{d}\right].$$

We write $[x/d]=x/d+O(1)$ and use that $\mu(d)\in\{-1,0,1\}$. This gives

$$\sum_{d\le x}\frac{\mu(d)}{d}\left[\frac{x}{d}\right]=\sum_{d\le x}\frac{\mu(d)}{d}\frac{x}{d}+O\left(\sum_{d\le x}\frac{1}{d}\right)=\sum_{d\le x}\frac{\mu(d)}{d}\frac{x}{d}+O(\log x)$$

(by approximating the n-th harmonic number with the logarithm) and we have

$$\sum_{d\le x}\frac{\mu(d)}{d}\frac{x}{d}=x\sum_{d=1}^{\infty}\mu(d)d^{-2}+O\left(x\sum_{x<d<\infty}d^{-2}\right)=x\zeta(2)^{-1}+O(1).$$

One can show the estimate $\sum_{x<d<\infty}d^{-2}\ll x^{-1}$ using the inequality

$$\sum_{x<d<\infty}d^{-2}\le\int_{x-1}^{\infty}t^{-2}\,\mathrm{d}t=O((x-1)^{-1})=O(x^{-1}).$$

Done.

Notes after correcting.

- The convolution formula can also be obtained formally by writing

  $$\varphi(n)=\sum_{k\le n\text{ and }(k,n)=1}1=\sum_{k\le n}\sum_{d|(k,n)}\mu(d)$$

  and reordering sums.

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Prof. Dr. Valentin Blomer winter term 2022/23

Analytic Number Theory
(half) Problem Set 3

Problem 1 & 2. In this exercise we want to classify all *real* primitive Dirichlet characters. As a preparation, it's good to refresh your knowledge on the structure of $(\mathbb{Z}/n\mathbb{Z})^*$ and quadratic reciprocity.

a) Let p be an odd prime. Show that there is exactly one real primitive character modulo p and no real primitive character modulo p^r for any $r \geqslant 2$.

b) Show that there is no real primitive character modulo 2, exactly one real primitive character χ_{-4} modulo 4, exactly two real primitive characters χ_8, χ_{-8} modulo 8, and no real primitive character modulo 2^r, $r > 3$. Write down the values of χ_{-4}, χ_8 and χ_{-8}.

c) Let $n = rs$ with $(r, s) = 1$. Show that a Dirichlet character χ modulo n factors uniquely into a product of a Dirichlet character modulo r and a Dirichlet character modulo s. These two are primitive if and only if χ is primitive.

d) For $n \in \mathbb{N}$, describe all real primitive characters modulo n.

e) A fundamental discriminant is an integer D such that

$$D \equiv 1 \pmod 4, \quad D \text{ squarefree}$$

or

$$D \equiv 0 \pmod 4, \quad D/4 \text{ squarefree}, \quad D/4 \equiv 2 \text{ or } 3 \pmod 4.$$

(These are exactly the discriminants of quadratic extensions over $\mathbb{Q}$.) For a fundamental discriminant D define a function (Kronecker symbol) $\chi_D : \mathbb{Z} \to \{-1, 0, 1\}$ by

$$\chi_D(p) = \left(\frac{D}{p}\right), \quad p \text{ odd prime}$$

$$\chi_D(2) = \begin{cases} 0, & D \equiv 0 \pmod 4 \\ 1, & D \equiv 1 \pmod 8 \\ -1, & D \equiv 5 \pmod 8 \end{cases}$$

$$\chi_D(-1) = \operatorname{sgn}(D)$$

$$\chi_D(ab) = \chi_D(a)\chi_D(b) \quad \text{for all } a, b \in \mathbb{Z}.$$

Show that the real primitive characters are precisely the characters χ_D for D a fundamental discriminant.

Due: Fri, Oct 28

Solution to Sheet 3.

Facts from multiplicative number theory.

Given some $n = p_1^{e_1} \cdots p_r^{e_r} \in \mathbb{N}$, we want to investigate the structure of the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$. By the chinese remainder theorem we find

$$(\mathbb{Z}/n\mathbb{Z})^\times \cong \left(\prod_{i=1}^{n} (\mathbb{Z}/p_i^{e_i}\mathbb{Z}) \right)^\times \cong \prod_{i=1}^{n} (\mathbb{Z}/p_i^{e_i}\mathbb{Z})^\times,$$

so we really only care about the structure of $(\mathbb{Z}/p^e\mathbb{Z})^\times$. There, the structure is given by

$$(\mathbb{Z}/p^e\mathbb{Z})^\times \cong \begin{cases} \text{a cyclic subgroup of order } \varphi(p^e) & \text{if } p \text{ is odd} \\ \langle 3 \rangle & \text{if } p = 2 \text{ and } e \leq 2 \\ \pm\langle 5 \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{e-2}\mathbb{Z} & \text{if } p = 2 \text{ and } e \geq 3. \end{cases}$$

A generator of $\mathbb{F}_p^\times$, or more generally, a generator of $(\mathbb{Z}/p^e\mathbb{Z})^\times$ is called a *root of unity.* We have the *Legendre symbol*, which for $a \in \mathbb{Z}$ and an odd prime p is given by

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \mid a \\ (-1) & \text{if there is no solution mod } p \text{ to } x^2 = a \\ 1 & \text{otherwise.} \end{cases}$$

It is multiplicative in a, hence it yields a character $(\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{C}^\times$. The subgroup of *quadratic residues mod* p is given by $\operatorname{Ker}\left(\left(\frac{-}{p}\right)\right) = \langle \varpi^2 \rangle$ for ϖ a root of unity. *Quadratic reciprocity* states that for two odd primes p, q, we have

$$\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}} \left(\frac{q}{p}\right),$$

and there are the *supplementary laws*

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} \quad \text{and} \quad \left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}.$$

Given a finite abelian group G, we define the group of *characters of* G as

$$\widehat{G} = \operatorname{Hom}_{\mathsf{Ab}}(G, \mathbb{C}^\times) = \operatorname{Hom}_{\mathsf{Ab}}(G, S^1).$$

Given a cyclic group $G \cong \mathbb{Z}/n\mathbb{Z}$, there is an isomorphism $G \cong \widehat{G}$ given by $a \mapsto (1 \mapsto \zeta_n^a)$, where ζ_n is an n-th root of unity. As we also have $\widehat{G} \oplus \widehat{H} = \widehat{G \oplus H}$, this shows that there are isomorphisms $G \cong \widehat{G}$ for *all* finite abelian groups[1].

[1] The first isomorphism is the universal property of the direct sum: We have

$$\operatorname{Hom}_{\mathsf{Ab}}(G \oplus H, \mathbb{C}^\times) \cong \operatorname{Hom}_{\mathsf{Ab}}(G, \mathbb{C}^\times) \oplus \operatorname{Hom}_{\mathsf{Ab}}(H, \mathbb{C}^\times).$$

Remember that every finite group is a finite product (equivalently, finite direct sum) of cyclic groups.

Exercise 1 & 2.

1. Note that the real characters are exactly those $\chi : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{C}^\times$ with $\chi^2 = 1$. As p is odd, there are exactly two solutions to $x^2 = 1$, hence there are exactly 2 real characters mod p, one of which is the trivial one (induced by the principle character mod 1), and the other is given by the legendre symbol. The same reasoning goes through mod p^e for $e \geq 2$ (the multiplicative group is cyclic of even order), but now the characters are induced from characters mod p.

2. For $n = 2^r$, we find again that the real Dirichlet characters are in bijection with the set $\{x \in \mathbb{Z}/n\mathbb{Z} \mid x^2 - 1 = 0\}$. By the structure of the multiplicative group given above, this set has 1 element if $r = 1$, it has 2 elements if $r = 2$ and 4 elements if $r \geq 3$. We find:

- The multiplicative group of $\mathbb{Z}/2\mathbb{Z}$ is trivial, so there is only the character given by $1 \mapsto 1$, which is induced by the principle character.
- On $\mathbb{Z}/4\mathbb{Z}$ we have again the principle character and the primitive character χ_{-4} uniquely defined via $\chi_{-4}(-1) = -1$.
- On $\mathbb{Z}/8\mathbb{Z}$ we have the principle character, the one induced by χ_{-4} and the two characters $\chi_{\pm 8}$, where $\chi_{\pm 8}(3) = \mp 1$, $\chi_{\pm 8}(5) = -1$ and $\chi_{\pm 8}(7) = \pm 1$.

3. We inspect the map

$$\mu : \widehat{(\mathbb{Z}/r\mathbb{Z})^\times} \times \widehat{(\mathbb{Z}/s\mathbb{Z})^\times} \to \widehat{(\mathbb{Z}/n\mathbb{Z})^\times} \quad (\chi_1, \chi_2) \mapsto \chi_1\chi_2.$$

We claim that this map is injective. Indeed, assume that we are given two characters χ_1 mod r and χ_2 mod s such that for all $m \in \mathbb{N}$,

$$\chi(m) = \chi_1(m \bmod r)\chi_2(m \bmod s).$$

Then whenever we are given $m \in \mathbb{N}$ such that $m \equiv 1 \bmod s$, we find

$$\chi(m) = \chi_1(m),$$

and similarly for χ_2. But the chinese remainder theorem asserts that these equalities already define χ_1 and χ_2 uniquely: For any $a \in (\mathbb{Z}/r\mathbb{Z})^\times$, there is some $m \in \mathbb{N}$ such that $m \equiv a \mod r$ and $m \equiv 1 \mod s$. Now μ is an injective map of sets with the same cardinality, hence bijective.

It remains to show that χ_1 and χ_2 are primitive iff χ is. Suppose first that χ_1 was not primitive, i.e., has conductor $d < r$. Then we can write $\chi_1 = \tilde{\chi}\chi_{0,r}$ where $\tilde{\chi}$ is a character mod d and $\chi_{0,r}$ is the principal character mod r. Now $\chi' = \tilde{\chi}\chi_2$ is a character modulo ds and induces χ, since

$$\chi = \chi\chi_{0,rs} = \chi_1\chi_2\chi_{0,rs} = \tilde{\chi}\chi_{0,r}\chi_2\chi_{0,rs} = \chi'\chi_{0,r}\chi_{0,rs} = \chi'\chi_{0,rs}.$$

There is a neat way to now show the converse. Let $\varphi_2(n)$ denote the number of primitive characters mod n. For any $d \mid n$, the set of primitive characters mod d is in bijection with the characters mod n of conductor d, so we find

$$\varphi(n) = \#\widehat{(\mathbb{Z}/n\mathbb{Z})^\times} = \sum_{d|n} \varphi_2(n) = (1 \star \varphi_2)(n),$$

implying that $\varphi_2 = \mu \star \varphi$ by moebius-inversion. Hence φ_2 is multiplicative. We have shown alrady that the inverse of μ restricts to a (necessarily) injective map

$$\mu^{-1} : \{\text{primitive characters mod } n\} \to \{\text{pr. characters mod } r\} \times \{\text{pr. characters mod } s\}.$$

By multiplicity of φ_2, this is a injective map of sets of the same cardinality, therefore μ^{-1} is a bijection, and we are done.

Alternatively we can calculate this directly. Assume that χ_1 and χ_2 are primitive. Choose a character $\tilde{\chi}$ mod d that induces χ, so we may write

$$\chi_1\chi_2 = \tilde{\chi}\chi_{0,rs} = (\tilde{\chi}_1\chi_{0,r})(\tilde{\chi}_2\chi_{0,s}),$$

where $\tilde{\chi}_1$ is a character of conducter $d_1 \mid r$ and $\tilde{\chi}_2$ is a character of conducter $d_2 \mid s$. But by uniqueness of χ_1 and χ_2, we find $\chi_1 = \tilde{\chi}\chi_{0,r}$ and $\chi_2 = \tilde{\chi}\chi_{0,s}$, implying $d = rs$ by primitivity of χ_1 and χ_2.

4. Writing $n = 2^r q$ with q odd, we find that the number of primitive real characters mod n is given by

$$\begin{cases} 1 & \text{if } r = 0 \text{ and } q \text{ square-free}, \\ 0 & \text{if } r = 1 \text{ and } q \text{ square-free}, \\ 1 & \text{if } r = 2 \text{ and } q \text{ square-free}, \\ 2 & \text{if } r = 3 \text{ and } q \text{ square-free}, \\ 0 & \text{if } r \geq 4 \text{ or } q \text{ not square-free}. \end{cases}$$

5. Clearly the product of two fundamental discriminants (FDs) is again a FD, and we have $\chi_{D_1D_2} = \chi_{D_1}\chi_{D_2}$. Also, given a fundamental discriminant D with $|D| = d_1d_2$ and $(d_1, d_2) = 1$, there are fundamental discriminants D_1, D_2 with $d_i = \pm D_i$ and $D_1D_2 = D$. So we can reduce to the case where $|D| = p^r$ is a prime power. As a first reality check, we find that if p is odd, the only fundamental discriminant of this type is $D = (-1)^{\frac{p-1}{2}}p$, in which case χ_D is given by the unique real primitive character, given by (using quadratic reciprocity)

$$\chi_D(q) = \left(\frac{(-1)^{(p-1)/2}p}{q}\right) = \left(\frac{q}{p}\right).$$

There are no FDs with $|D| = 2$ or $|D| = 2^r$ with $r \geq 4$. If $|D| = 4$ there is one ($D = -4$), and if $n = 8$ there are two ($D = \pm 8$). Using quadratic reciprocity and the supplementary laws, it is easily seen that these are exactly the characters described above.

Prof. Dr. Valentin Blomer winter term 2022/23

Analytic Number Theory
Problem Set 4

Problem 1. Prove Corollary (2.7).

Problem 2. Prove Lemma (2.9). *Hint:* Consider functions of the type $f(x) = \begin{cases} e^{-1/x^2}, & x > 0 \\ 0, & x \leqslant 0 \end{cases}$ and $g(x) = f(x)/(f(x) + f(1-x))$.

Problem 3. Show that the Polya-Vinogradov inequality is essentially optimal: for a primitive character χ modulo q one has

$$\max_{x \geqslant 1} \left| \sum_{n \leqslant x} \chi(n) \right| \geqslant \frac{1}{2\pi} \sqrt{q}.$$

Hint: Apply partial summation to the definition of the Gauss sum $\tau(\chi)$ (two lines).

Problem 4. Let $(q_1, q_2) = 1$ and let χ_1 modulo q_1 and χ_2 modulo q_2 be two Dirichlet characters (not necessarily primitive). Show that $\tau(\chi_1\chi_2) = \tau(\chi_1)\tau(\chi_2)\chi_1(q_2)\chi_2(q_1)$.
Hint: Chinese remainder theorem

Due: Tue, Nov 8

Solution to Sheet 4.

Problem 1

a) Let $g(x) = f(qx + a)$, so that

$$\sum_{n \equiv a \pmod q} f(n) = \sum_{m \in \mathbb{Z}} g(m).$$

We want to apply Poisson summation to g. The results of lemma (2.3) directly give that

$$\hat{g}(y) = \frac{1}{q} e\left(\frac{ya}{q}\right) \hat{f}\left(\frac{y}{q}\right).$$

The claim follows, as

$$\sum_{m \in \mathbb{Z}} g(m) = \sum_{m \in \mathbb{Z}} \hat{g}(m) = \frac{1}{q} \sum_{m \in \mathbb{Z}} e\left(\frac{ma}{q}\right) \hat{f}\left(\frac{m}{q}\right).$$

b) We would like to apply Poisson summation again, however we cannot calculate the "Fourier transform" of $f\chi$, as, χ is only defined on integers. We can abuse that χ is periodic though, rewriting

$$\sum_{m \in \mathbb{Z}} f(m)\chi(m) = \sum_{a \pmod q} \chi(a) \sum_{m \equiv a \pmod q} f(m).$$

Applying Poisson summation to the inner sum (we already did this in part a)) gives

$$\sum_{m \in \mathbb{Z}} f(m)\chi(m) = \frac{1}{q} \sum_{a \pmod q} \chi(a) \sum_{m \in \mathbb{Z}} e\left(\frac{ma}{q}\right) \hat{f}\left(\frac{m}{q}\right).$$

Reordering sums, we obtain

$$\begin{aligned}\frac{1}{q} \sum_{a \pmod q} \chi(a) \sum_{m \in \mathbb{Z}} e\left(\frac{ma}{q}\right) \hat{f}\left(\frac{m}{q}\right) &= \frac{1}{q} \sum_{m \in \mathbb{Z}} \hat{f}\left(\frac{m}{q}\right) \left(\sum_{a \pmod q} \chi(a) e\left(\frac{ma}{q}\right)\right) \\ &= \frac{1}{q} \sum_{m \in \mathbb{Z}} \hat{f}\left(\frac{m}{q}\right) \tau(\chi)\overline{\chi}(m) = \frac{\tau(\chi)}{q} \sum_{m \in \mathbb{Z}} \hat{f}\left(\frac{m}{q}\right) \overline{\chi}(m).\end{aligned}$$

Notes after correcting.

- In part a), instead of using the results from the lecture, we can also obtain the formula for the fourier transform directly. Setting $g(x) = f(qx + a)$ and substituting $u = qx + a$, we obtain

$$\hat{g}(y) = \int_{\mathbb{R}} f(qx + a) e(-xy) \, \mathrm{d}x = \frac{1}{q} \int_{\mathbb{R}} f(u) e(-u\tfrac{y}{q} + \tfrac{ay}{q}) \, \mathrm{d}u = \frac{1}{q} e(\tfrac{ay}{q}) \hat{f}(\tfrac{y}{q}).$$

Problem 2

We do as the hint commands. Let

$$f(t) = \begin{cases} \mathrm{e}^{-1/t^2} & t > 0 \\ 0 & \text{else.} \end{cases}$$

Then one easily checks that f is smooth and non-negative. Now we put $g(t) = \frac{f(t)}{f(t)+f(1-t)}$, which is still smooth and non-negative. We clearly have $g(t) = 0$ if $t < 0$, $g(t) \in [0, 1]$ for $t \in [0, 1]$ and $g(t) = 1$ for $t > 1$. Finally, define

$$h(t) = g\left(\frac{t - X + Z}{Z}\right) - g\left(\frac{t - X - Y}{Z}\right).$$

This satisfies $\operatorname{supp}(h) \subset [X - Z, X + Y + Z]$ and $h(t) = 1$ for $t \in [X, X + Y]$. We still need to check that $\|f^{(j)}\|_1 \ll Z^{1-j}$ for all $j \in \mathbb{N}$. One could expect this to be really messy as calculating the higher derivatives of h seems horrible. However, we just need that the j-th derivative of h is given by

$$h^{(j)}(t) = Z^{-j}\left(g^{(j)}\left(\tfrac{t-X+Z}{Z}\right) - g^{(j)}\left(\tfrac{t-X-Y}{Z}\right)\right).$$

As $h^{(j)}$ vanishes everywhere except $[X - Z, X]$ and $[X + Y, X + Y + Z]$, we obtain by a linear change of variables

$$\|h^{(j)}\|_1 = \left(\int_{X+Z}^{X} + \int_{X+Y}^{X+Y+Z}\right)\left|h^{(j)}(t)\right| \,\mathrm{d}t = 2Z^{1-j}\int_0^1 \left|g^{(j)}(t)\right| \,\mathrm{d}t \ll_j Z^{1-j}.$$

Problem 3

As the hint commands, we apply partial summation to the definition of $\tau(\chi)$, obtaining

$$|\tau(\chi)| = \sum_{h=1}^{q} \chi(h)e(h/q) = e(q/q)\sum_{h=1}^{q}\chi(h) - \frac{2\pi \mathrm{i}}{q}\int_1^q e(t/q)\sum_{h\le t}\chi(h)\,\mathrm{d}t.$$

As $\chi \neq \chi_0$, the sum $\sum_{h=1}^{q}\chi(h)$ vanishes. We also know by theorem (1.23) that $|\tau(\chi)| = \sqrt{q}$. Let M deonte the supremum of the absolute values of $\sum_{h\le x}\chi(h)$ for varying x (By Polya-Vinogradov, $M < \infty$). Then we obtain

$$\frac{q^{3/2}}{2\pi} = \left|\int_1^q e(t/q)\sum_{h\le t}\chi(h)\,\mathrm{d}t\right| \le \int_1^q \left|\sum_{h\le t}\chi(h)\right| \,\mathrm{d}t \le (q-1)M,$$

which is even a tad stronger than what we had to show.

Problem 4

Let's just plug in the definition and look at what we have here.

$$\tau(\chi_1\chi_2) = \sum_{h\ (q)} \chi_1(h)\chi_2(h)e(h/q),$$

where $q = q_1 q_2$. By the chinese remainder theorem, taking residues mod q gives a bijection

$$\{h_1 q_2 + h_2 q_1 \mid 1 \leq h_i \leq q_i\} \to \mathbb{Z}/q\mathbb{Z}.$$

Thus we may rewrite the sum above as

$$\tau(\chi_1\chi_2) = \sum_{1 \leq h_1 \leq q_1} \sum_{1 \leq h_2 \leq q_2} \chi_1(h_1 q_2 + h_2 q_1)\chi_2(h_1 q_2 + h_2 q_1) e(\tfrac{h_1 q_2 + h_2 q_1}{q}),$$

and the claim follows after a few manipulations:

$$\begin{aligned}
&\sum_{1 \leq h_1 \leq q_1} \sum_{1 \leq h_2 \leq q_2} \chi_1(h_1 q_2 + h_2 q_1)\chi_2(h_1 q_2 + h_2 q_1) e(\tfrac{h_1 q_2 + h_2 q_1}{q}) \\
&\qquad = \sum_{1 \leq h_1 \leq q_1} \sum_{1 \leq h_2 \leq q_2} \chi_1(h_1 q_2)\chi_2(h_2 q_1) e(\tfrac{h_1 q_2}{q}) e(\tfrac{h_2 q_1}{q}) \\
&= \left(\chi_1(q_2) \sum_{1 \leq h_1 \leq q_1} \chi_1(q_2) e(\tfrac{h_1}{q_1})\right) \left(\chi_2(q_1) \sum_{1 \leq h_2 \leq q_2} \chi_2(q_1) e(\tfrac{h_2}{q_2})\right) = \chi_1(q_2)\tau(\chi_1)\chi_2(q_1)\tau(\chi_2).
\end{aligned}$$

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Prof. Dr. Valentin Blomer winter term 2022/23

Analytic Number Theory

$\frac{3}{4}$ Problem Set 5

Problem 1. Let χ_1 modulo q_1 and χ_2 modulo q_2 be two Dirichlet characters with conductors d_1, d_2. Assume that $(q_1, q_2) = 1$. Show that the conductor of $\chi_1\chi_2$ is $d_1 d_2$. Is this correct without the coprimality assumption?

Problem 2. For $A \in \mathbb{N}$, $x > 0$ show that

$$\int_{\Re s = -A + \frac{1}{2}} \Gamma(s)\, x^s ds \ll \frac{x^{-A+1/2}}{(A-1)!}$$

where the implied constant is absolute (independent of A and x). *Hint:* Stirling's formula cannot be applied immediately (why?). Use the recurrence relation of the Gamma-function A times.

Problem 3. Let α_j, $1 \leqslant j \leqslant d$ be distinct complex numbers, p a prime, and for $\Re s$ sufficiently large write

$$\prod_{j=1}^{d} \left(1 - \frac{\alpha_j}{p^s}\right)^{-1} = \sum_{k=0}^{\infty} \frac{\beta(k)}{p^{ks}}.$$

Show that $\beta(k)$ satisfies a d-term recurrence, i.e. there are d complex numbers $c_0, \ldots, c_{d-1}$ such that $c_0\beta(\nu) + \ldots + c_{d-1}\beta(\nu + d - 1) = \beta(\nu + d)$ for all $\nu \in \mathbb{N}$. *Hint:* you see more if you write $x = p^{-s}$.

Due: Tue, Nov 15

Solution to Sheet 5.

Problem 1

We have basically solved this already on sheet 3. Note that as $d_1 \mid q_1$ and $d_2 \mid q_2$, we have $(d_1, d_2) = 1$, so (by sheet 3) there are primitive characters ψ_i mod d_i with $\chi_i = \psi_i \chi_{0,q_i}$ (here again χ_{0,q_i} is the principal character mod q_i) whose product $\psi = \psi_1\psi_2$ is a primitive character mod $d_1 d_2$. Modulo q, this reveals

$$\chi_1\chi_2 = (\chi_{0,q_1}\psi_1)(\chi_{0,q_2}\psi_2) = \chi_{0,q_1 q_2}\psi,$$

hence ψ is induced by a primitive character mod $d_1 d_2$.

It is easily seen that the coprimality condition is necessary. Take any real character χ mod q for example, then $\chi^2 = 1$ and has conducter $1 \neq q$.

Problem 2

We have to show the bound

$$\int_{(-A+\frac{1}{2})} \Gamma(s) x^s \,\mathrm{d}s \ll \frac{x^{-A+1/2}}{(A-1)!}.$$

Note that the integral exists by the rapid decay of Γ along vertical lines. However, we cannot apply Stirling's formula to bound the integral directly as Stirling a priori only gives uniform bounds in regions of the form $|\arg(s) - \pi| \geq \delta > 0$. We can however apply stirlings formula if we apply the recurrence $s\Gamma(s) = \Gamma(s+1)$ repeatedly:

$$\begin{aligned}\int_{(-A+1/2)} \Gamma(s) x^s \,\mathrm{d}s &\ll \int_{(-A+1/2)} |\Gamma(s) x^s| \,\mathrm{d}s \ll x^{-A+1/2} \int_{(1/2)} |\Gamma(s-A)| \,\mathrm{d}s \\ &= x^{-A+1/2} \int_{(1/2)} \left| \frac{\Gamma(s)}{(s-A+1)\cdots(s-1)} \right| \mathrm{d}s \leq \frac{x^{-A+1/2}}{(A-1)!} \int_{(1/2)} |\Gamma(s)| \,\mathrm{d}s.\end{aligned}$$

Notes. Once we know this inequality, we actually can do better: Remember that Γ has poles at the negative integers, the residue at $-n$ is given by $\frac{(-1)^n}{n!}$. Hence for (large) $T > 0$, we have that

$$\int_{1/2-A-\mathrm{i}T}^{1/2-A+\mathrm{i}T} \Gamma(s) x^s \,\mathrm{d}s = 2\pi\mathrm{i} \frac{(-x)^{-A}}{A!} + \int_{1/2-A-\mathrm{i}T}^{-1/2-A+\mathrm{i}T} \Gamma(s) x^s \,\mathrm{d}s + O\left(\int_{1/2-A-\mathrm{i}T}^{-1/2-A-\mathrm{i}T} \Gamma(s) x^s \,\mathrm{d}s \right).$$

By the rapid decay of Γ, the horizontal integral vanishes as $T \to \infty$, and we can bound the vertical integral using what we showed before, applied to $A+1$. This yields

$$\int_{(-A+1/2)} \Gamma(s) x^s \,\mathrm{d}s = 2\pi\mathrm{i} \frac{(-x)^{-A}}{A!} + O\left(\frac{x^{-A-1/2}}{A!} \right).$$

In fact, as for every $x > 0$ the fraction $x^A/A!$ tends to zero as $A \to \infty$, we may repeat this as often as we want, obtaining

$$\frac{1}{2\pi\mathrm{i}} \int_{(-A+1/2)} \Gamma(s) x^s \,\mathrm{d}s = \sum_{k=A}^{\infty} \frac{(-x)^{-k}}{k!} = \mathrm{e}^{-\frac{1}{x}} - \sum_{k=0}^{A-1} \frac{(-x)^{-k}}{k!}.$$

The equation for $A = 0$ is nothing new! As $\Gamma(s)$ is holomorphic for $\Re s > 0$ we already know that $\mathcal{M}(e^{-x})(s) = \Gamma(s)$, so

$$\mathrm{e}^{-x} = \frac{1}{2\pi\mathrm{i}} \int_{(1/2)} \mathcal{M}(\mathrm{e}^{-x})(s) x^{-s} \, \mathrm{d}s = \frac{1}{2\pi\mathrm{i}} \int_{(1/2)} \Gamma(s) x^{-s} \, \mathrm{d}s.$$

Now replace x by x^{-1}.

Problem 3

We substitute $p^{-s} = x$ to find the equivalent

$$\sum_{k=0}^{\infty} \beta(k) x^k = \frac{1}{P(x)}.$$

Where $P(x) = \prod_{j=1}^{d}(1 - \alpha_j x) = \sum_{i=0}^{d} a_i x^i$ (in particular, $a_0 = 1$). Multiply both sides with P, revealing

$$\sum_{d=0}^{\infty} x^d \sum_{k=0}^{d} \beta(d-k) a_k = 1.$$

Equating coeffiecients gives that for $k > 0$,

$$\sum_{k=0}^{d} a_k \beta(d-k) = 0,$$

which, after subtracting $\beta(d)$ on both sides and setting $c_i = -a_{i+1}$, gives the desired recurrence.

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Prof. Dr. Valentin Blomer winter term 2022/23

Analytic Number Theory
Problem Set 6

Problem 1. Write down the statements of (3.10) - (3.12) in the special case of the Riemann zeta function an Dirichlet L-functions for primitive characters in full detail.

Problem 2. Compute $\zeta(0)$ and show that $\zeta(s)$ has no zeros on the real segment $[0, 1)$. *Hint for the second part:* (1.9a).

Problems 3.-4. Show

$$\sum_{\chi \,(\mathrm{mod}\; q)} |L(1/2, \chi)|^4 \ll q^{1+\varepsilon}.$$

Where does this argument go wrong for the sixth moment?
Hint: Restrict to primitive characters of modulus $q_1 \mid q$ and estimate for each q_1 separately. Use (3.14) as a template. You have now four variables n_1, n_2, m_1, m_2, say, each of effective length $q_1^{1/2}$. Glue together two pairs of variables n_1, n_2, m_1, m_2 to two new variables n, m of effective length at most q_1, each of which is weighted by a divisor function.

Due: Tue, Nov 22

Solutions to Sheet 6.

Problem 1

Okay, we just go through everyting. For $\zeta(s)$ we have degree $d = 1$, conductor $N = 1$, root number $\eta = 1$, $\kappa_1 = 0$ and hence $L_\infty(s) = \pi^{-s/2}\Gamma(s/2)$. For $L(s,\chi)$ with a primitive Dirichlet character χ mod $q > 1$ we have degree $d = 1$, conductor $N = q$, root number $\eta = \mathrm{i}^{-\kappa}\tau(\chi)q^{-1/2}$, $\kappa_1 = \kappa$ and $L_\infty(s) = \pi^{-s/2}\Gamma(\frac{s+\kappa}{2})$ where $\kappa = 0$ if χ is even and $\kappa = 1$ if χ is odd.

The functional equation now reads as follows.

Theorem 1 (Approximate functional equation for ζ)**.** *Let $G(u)$ be any even function which is holomorphic and bounded in $|\Re(u)| < 4$ and normalized by $G(0) = 1$. Let $X > 0$. Then for $0 < \sigma < 1$ we have*

$$\zeta(s) = \sum_n n^{-s} V_s\left(\frac{n}{X}\right) + \pi^{s-1/2}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}\sum_n n^{s-1}V_{1-s}(nX) - R$$

where

$$V_s(y) = \frac{1}{2\pi \mathrm{i}}\int_{(3)} G(u)\frac{\Gamma((s+u)/2)}{\Gamma(s/2)}(y\sqrt{\pi})^{-u}\frac{\mathrm{d}u}{u}$$

and

$$R = \frac{\pi^{s/2}}{\Gamma(s/2)}\frac{G(1-s)}{1-s}X^{1-s} - \frac{\pi^{s/2}}{\Gamma(s/2)}\frac{G(-s)}{-s}X^{-s}.$$

Completed Dirichlet L-functions are entire, so we get rid of R. As above, in the following κ depends on the parity of χ.

Theorem 2 (Approximate functional equation for Dirichlet L-functions)**.** *Let $G(u)$ be any even function which is holomorphic and bounded in $|\Re(u)| < 4$ and normalized by $G(0) = 1$. Let $X > 0$. Then for $0 < \sigma < 1$ we have*

$$\zeta(s) = \sum_n \chi(n) n^{-s} V_s\left(\frac{n}{X\sqrt{q}}\right) + \epsilon(s)\sum_n \overline{\chi(n)} n^{s-1}V_{1-s}\left(\frac{nX}{\sqrt{q}}\right)$$

where

$$V_s(y) = \frac{1}{2\pi \mathrm{i}}\int_{(3)} G(u)\frac{\Gamma((s+u+\kappa)/2)}{\Gamma((s+\kappa)/2)}(y\sqrt{\pi})^{-u}\frac{\mathrm{d}u}{u}$$

and

$$\epsilon(s) = \mathrm{i}^{-\kappa}\tau(\chi)q^{-s}\pi^{s-1/2}\frac{\Gamma((1-s+\kappa)/2)}{\Gamma((s+\kappa)2)}.$$

As for (3.11), we have for ζ that $\mathcal{C}(s) = \mathcal{C}_0(s) = |s+2|$, for Dirichlet L-functions we find $\mathcal{C}_0(s) = |s+\kappa| + 2$ and $\mathcal{C}(s) = q(|s+\kappa|+2)$. As an aside, the 2 here is quite arbitrary and is only there to make sure everything works out when $|s|$ is small. We can plug this into (3.11), finding (with $G(u) = \mathrm{e}^{u^2}$) that for ζ we have that

$$y^a V_s^{(a)}(y) \ll_{a,A} \left(1 + \frac{y}{\sqrt{|s|+2}}\right)^{-A}$$

for $\Re(s) > 0$ whereas for $L(s,\chi)$ we find

$$y^a V_s^{(a)}(y) \ll_{a,A} \left(1 + \frac{y}{\sqrt{|s+\kappa|+2}}\right)^{-A}$$

for $\Re(s) > -\kappa$.

Lastly, the conditions for (3.12) are satisfied for both $\zeta(s)$ and $L(s,\chi)$, we have the convexity bound

$$\zeta(s) \ll_{\varepsilon,\delta} (|s|+2)^{\frac{1-\sigma}{2}+\varepsilon}$$

whenever $|s-1| \geq \delta$ (i.e., away from the pole) and similarly

$$L(s,\chi) \ll_{\varepsilon} (q\,|s+\kappa|+2)^{\frac{1-\sigma}{2}+\varepsilon}.$$

Again, it should be noted that the 2 is added artificially to have small $|s|$ not mess everything up. For large s, these vanish and we obtain (and should really read these as)

$$\zeta(s) \ll |s|^{\frac{1-\sigma}{2}+\varepsilon} \quad \text{and} \quad L(s,\chi) \ll |qs|^{\frac{1-\sigma}{2}+\varepsilon}.$$

Also, if we fix L, we can absorb the factor q into the implicit constant from $\ll$.

Problem 2

1. *Calculating* $\zeta(0)$. The simple pole of $\zeta(s)$ at $s=1$ has residue 1, so we know that $\lim_{s\to 1}(s-1)\zeta(s) = 1$. Writing the functional equation as $\zeta(s) = \Delta(s)\zeta(1-s)$ gives

$$1 = \lim_{s\to 1}(s-1)\zeta(s) = \lim_{s\to 1}(s-1)\Delta(s)\zeta(0),$$

so we only need to evaluate the remaining term $\lim_{s\to 1}(s-1)\Delta(s)$. We have

$$\Delta(s) = \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\pi^{s-1/2}.$$

It follows that $\zeta(0) = -\frac{1}{2}$ as $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma((1-s)/2)$ has residue -2 at 1 (think about the Laurent expansion at 1 and remember that Γ has residue 1 at 0).

2. Showing that $\zeta(s) < 0$ *for* $s \in (0,1)$. We have that

$$\zeta(s) = \frac{s}{s-1} - s\int_0^\infty \{t\} t^{-s-1}\,\mathrm{d}t.$$

This is negative. Hence ζ is negative in the interval $[0,1)$.

Problem 3

We want to follow the proof from (3.14) as closely as possible. The first difference is that we sum over all characters, not just the primitive ones, but this does not make a difference: If we know that

$$\sum_{\chi (\mathrm{mod}\ q)}^{*} |L(1/2,\chi)|^2 \ll q^{1+\varepsilon}$$

(where the star in the sum means that we sum over primitive characters, this notation is quite common), we can easily deduce

$$\sum_{\chi (\text{mod } q)} |L(1/2,\chi)|^2 = \sum_{d|q} \sum_{\chi (\text{mod } d)}^{*} |L(1/2,\chi)|^2 \leq \tau(q) \max_{d|q} \sum_{\chi (\text{mod } d)}^{*} |L(1/2,\chi)|^2 \ll q^{1+\varepsilon}$$

as $\tau(q)$ the number of divisors of q, satisfies $\tau(q) \ll q^{\varepsilon}$. The L_∞ factor occuring in V_s only depends on the parity of χ, so we further split the sum into odd and even parts. We want to use the approximate functional equation with $X = 1$, $s = 1/2$ and $G(u) = \mathrm{e}^{u^2}$ as in (3.11). Let's check what happens. We find

$$L(1/2,\chi) = \sum_n \frac{\chi(n)}{n^{1/2}} V_{1/2}(n/\sqrt{N}) + \epsilon(1/2) \sum_n \frac{\overline{\chi}(n)}{n^{1/2}} V_{1/2}(n/\sqrt{N}) + R$$

where

- $R = 0$ as the completed L-function $\Lambda(s,\chi)$ is entire.
- The root number $\epsilon(1/2)$ has absolute value 1.
- The terms involving $V = V_{1/2}$ can be bounded by $V(y) \ll_A (1+y)^{-A}$. For all $A > 0$.

Also note that both sums are equal in absolute value. This is not too complicated! We plug it in, using this time that $|a+b|^4 \leq 8(|a|^4 + |b|^4)$ (this can be seen using Hölder's inequality for example), obtaining

$$\sum_{\chi (\text{mod q}) \text{ even}}^{*} |L(1/2,\chi)|^4 \leq 16 \sum_{\chi(q) \text{ even}}^{*} \left| \sum_n \frac{\chi(n)}{n^{1/2}} V(n/\sqrt{q}) \right|^4.$$

Similar to the proof of (3.14), we can complete the sum to go over all characters and open up the sum, obtaining a fourfold sum which we can simplify using orthogonality relations on sums over characters. In short, we get

$$\cdots \leq 16 \sum_{\chi(q)} \left| \sum_n \frac{\chi(n)}{n^{1/2}} V(n/\sqrt{q}) \right|^4 = 16 \sum_{n_1,n_2,m_1,m_2} \frac{V_{n_1} V_{n_2} \overline{V}_{m_1} \overline{V}_{m_2}}{(n_1 n_2 m_1 m_2)^{1/2}} \sum_{\chi(q)} \chi(n_1+n_2-m_1-m_2), \quad (1)$$

where we wrote $V_n = V(n/\sqrt{q})$. The sum over χ does not vanish iff $n_1 n_2 \equiv m_1 m_2 \pmod{q}$, where it equals $\varphi(q)$. As the hint suggests, we glue together n_1 and n_2, m_1 and m_2, which leaves us with the taks of bounding terms of the form

$$(V * V)(n) = \sum_{n_1 n_2 = n} V_{n_1} V_{n_2}.$$

We find for any $A \geq 1$

$$(V * V)(n) \ll \sum_{n_1 n_2 = n} \left(1 + \frac{n_1}{\sqrt{q}}\right)^{-A} \left(1 + \frac{n_2}{\sqrt{q}}\right)^{-A} \leq \sum_{n_1 n_2 = n} \left(1 + \frac{n}{q}\right)^{-A} \ll_\varepsilon n^{\varepsilon} \left(1 + \frac{n}{q}\right)^{-A}.$$

With $A = 1 + \varepsilon$ we calculate

$$(1) \ll \varphi(q) \sum_{n,m} \frac{(V \star V)(n)\overline{(V \star V)(m)}}{(nm)^{1/2}} \ll \varphi(q) \sum_n \sum_{n \equiv m \geq n} (1 + \tfrac{m}{q})^{-1} (1 + \tfrac{n}{q})^{-1} (mn)^{-1/2}, \quad (2)$$

and upon applying the bound $\varphi(q) < q$ this is exactly the sum that arises in the end of the proof of (3.14)! (I might add lines on how to bound this once I have time).

Max von Consbruch, email: s6mavonc@uni-bonn.de. Date: December 1, 2022

Prof. Dr. Valentin Blomer winter term 2022/23

Analytic Number Theory
Problem Set 7

Problem 1. Prove Lemma 4.4.

Problem 2. - 3. a) Prove an asymptotic formula for

$$\sum_{\substack{n \text{ squarefull} \\ n \leqslant x}} 1$$

with error term $O(x^{1/4+\varepsilon})$. *Hints:* See Set 2, Problem 1. When you use Perron, shift to $\Re s = 1/4$ (picking up two poles!) and cut the vertical integral into dyadic pieces. Then use Cauchy-Schwarz and (3.16).

b) Prove an asymptotic formula for

$$\sum_{n \text{ squarefull}} e^{-n/x}$$

with error term $O(x^{1/6+\varepsilon})$. [The moral of this is: when you count with a smooth weight, you get better error terms.]

Problem 4. Let $a(n)$ denote the number of isomorphy classes of finite abelian groups of order n.

a) Show that $n \mapsto a(n)$ is multiplicative, and show that $a(p^r)$ is the number of partitions $r = r_1 + r_2 + \dots$ with $r_1 \geqslant r_2 \geqslant \dots > 0$ for a prime p. Conclude that

$$\sum_n a(n) n^{-s} = \zeta(s)\zeta(2s)\zeta(3s)\cdots$$

for $\Re s > 1$. Show in particular that the infinite product is aboslutely convergent.

b) Sketch a proof of

$$\sum_{n \leqslant x} a(n) \sim Cx, \quad C = \zeta(2)\zeta(3)\zeta(4)\cdots \approx 2.2948\dots$$

for $x \to \infty$. In particular, on average there are about 2.3 abelian group of given order. (voluntarily:) Can you obtain an asymptotic formula with an explicit error term?

Due: Tue, Nov 29

Solutions to Sheet 7.

Problem 1

a - 2p) We have $g(x) \ll x^{-(-u+av)}$ as $x \to 0$ and $g(x) \ll x^{-(-u+bv)}$ as $x \to \infty$. Hence in $-u+av < \mathrm{Re}(s) < -u+bv$ the mellin transform $\widehat{g}$ exists and is given by

$$\widehat{g}(s) = \int_0^\infty x^u f(x^v) x^s \frac{\mathrm{d}x}{x} = v^{-1} \int_0^\infty f(y) y^{(s+u)/v-1} \,\mathrm{d}y = v^{-1} f\left(\frac{s+u}{v}\right)$$

Now the RHS defines a holomorphic function in $-u+a'v < \mathrm{Re}(s) < -u+b'v$.

b - 3p) Of course, knowing bounds for f does not imply any bounds for f'. But knowing that we can derive f, we can make use of partial integration. We have

$$\int_0^\infty f(x)x^{s-1}\,\mathrm{d}x = \left[f(x)\frac{x^s}{s}\right]_0^\infty - \frac{1}{s}\int_0^\infty f'(x)x^s\,\mathrm{d}x$$

By assumption, the boundary terms vanish for $a < \mathrm{Re}(s) < b$, and the integral on the RHS exists (if this is not clear, try to first approximate the integrals by truncated ones from $1/T$ to T and let $T \to \infty$). Hence $\widehat{g}$ (with $g = f'$) exists in $a+1 < \mathrm{Re}(s) < b+1$ (note the shift $s \mapsto s+1$ in the integral). Same argument as before gives continuation of $\widehat{g}$ to $a'+1 < \mathrm{Re}\, s < b'+1$.

c - 3p) By assumption f has compact support, so the Mellin Transform exists everywhere and the same holds for the derivatives. We make use of what we showed in *b*) repeatedly, obtaining

$$\widehat{f}(s) = \frac{(-1)^N}{s(s+1)\dots(s+N-1)}\widehat{g}(s+N) = (-1)^N\frac{\Gamma(s)}{\Gamma(s+N)}\widehat{g}(s+N)$$

where $g = f^{(N)}$. The first Γ-factor behaves (for fixed real part and large imaginary part of s) like $O(|s|^{-N})$, so it remains to show that $\widehat{g}(s)$ is bounded with $\mathrm{Im}\, s \to \infty$. But the integral from the mellin transform can be bounded in absolute values, as

$$|g(s)| \leq \int_0^\infty \left|g(x)x^{s-1}\right|\,\mathrm{d}x \ll \int |g(x)|\, x^{\mathrm{Re}(s)-1}\,\mathrm{d}x.$$

This is convergent, and independent of $\mathrm{Im}(s)$.

d - 2p) Calculation:

$$\widehat{f \star h}(s) = \int_0^\infty (f\star h)(x)x^{s-1}\,\mathrm{d}x = \int_0^\infty\int_0^\infty f(t)h(x/t)t^{-1}\,\mathrm{d}t x^{s-1}\,\mathrm{d}x$$

$$= \int_0^\infty f(t)h(y)t^{s-1}y^{s-1}\,\mathrm{d}t\,\mathrm{d}y,$$

as desired. We made use of the substitution $y = x/t$, i.e. $\mathrm{d}y = t^{-1}\,\mathrm{d}x$.

Problem 2&3

a - 15p) We want to apply Perron. Remember that we showed earlier that the Dirichlet series attached to the characteristic function on the set of squarefull numbers is given by $\frac{\zeta(2s)\zeta(3s)}{\zeta(6s)}$.

Just as in one of the examples from the lecture, we apply Perron with $c = 1 + 1/\log x$ and $T = x^\alpha$ for some fixed $\alpha \in (0,1)$. The absolute value of the coefficients is ≤ 1 and we obtain

$$\sum_{n \leq x \text{ sqfull}} 1 = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}T}^{c+\mathrm{i}T} \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)} x^s \frac{\mathrm{d}s}{s} + O(T^{-1} x \log x).$$

We want to shift the contour to the left and pick up residues along the way. The most important tool to bound the vertical contribution is the moment bound, and this requires the real part of the argument to be at least $\frac{1}{2}$. Hence we shift to $\operatorname{Re} s = \frac{1}{4}$. The factor $\zeta^{-1}(6s)$ is still holomorphic here, so we only pick up the residues from $\zeta(2s)$ and $\zeta(3s)$. We obtain

$$\sum_{n \leq x \text{ sqfull}} 1 =$$

$$\frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + \left(\int_{c-\mathrm{i}T}^{1/4-\mathrm{i}T} + \int_{1/4-\mathrm{i}T}^{1/4+\mathrm{i}T} + \int_{1/4+\mathrm{i}T}^{c+\mathrm{i}T} \right) \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)} x^s \frac{\mathrm{d}s}{s} + O(T^{-1} x \log x).$$

First, note that $\zeta^{-1}(s)$ is bounded in $\operatorname{Re} s > 1 + \delta$, as

$$\left|\zeta^{-1}(s)\right| = \prod_p |1 - p^{-s}| \leq \prod_p (1 + p^{-1-\delta}) = \frac{\zeta(2+2\delta)}{\zeta(1+\delta)} \ll_\delta 1.$$

So we disregard this factor from now on. Let us first start with the vertical part. Here we have $|x^s| = x^{1/4}$, so the contribution is bounded by

$$\ll x^{1/4} \int_0^T \frac{|\zeta(1/2 + 2\mathrm{i}t)\zeta(3/4 + 3\mathrm{i}t)|}{1/4 + \mathrm{i}t} \, \mathrm{d}t.$$

We prove that the integral is bounded by x^ε. By splitting the integral into $\log x$ dyadic pieces $[M, 2M]$ for $M < T$. It suffices to show that

$$\int_M^{2M} \frac{|\zeta(1/2 + 2\mathrm{i}t)\zeta(3/4 + 3\mathrm{i}t)|}{1/4 + \mathrm{i}t} \, \mathrm{d}t \ll M^{1+\varepsilon}.$$

The denumerator is (throughout) of size $\gg M$, so we really only need to show that

$$\int_M^{2M} |\zeta(1/2 + 2\mathrm{i}t)\zeta(3/4 + 3\mathrm{i}t)| \, \mathrm{d}t \ll M^\varepsilon \ll T^\varepsilon$$

This is an immediate consequence of Cauchy-Schwartz and the moment bounds. Hence we can conclude

$$\int_0^T \frac{|\zeta(1/2 + 2\mathrm{i}t)\zeta(3/4 + 3\mathrm{i}t)|}{1/4 + \mathrm{i}t} \, \mathrm{d}t$$
$$\leq \left(\int_0^1 + \int_1^2 + \cdots + \int_{2^{\lfloor \log_2(T) \rfloor}}^{2^{\lfloor \log_2(T) \rfloor + 1}} \right) \frac{|\zeta(1/2 + 2\mathrm{i}t)\zeta(3/4 + 3\mathrm{i}t)|}{1/4 + \mathrm{i}t} \, \mathrm{d}t \ll \log_2(T) T^\varepsilon \ll T^\varepsilon.$$

Next, we focus on the horizontal parts. Here, $s^{-1} \ll T^{-1}$, so the contributions become

$$\ll T^{-1} \int_{1/4}^{c} |\zeta(2(\sigma + \mathrm{i}T))\zeta(3(\sigma + \mathrm{i}T)| \, \mathrm{d}\sigma \ll T^{-1} \int_{1/4}^{c} T^{\max(1/2-\sigma,0)} T^{\max(1/2-3\sigma/2,0)} x^\sigma \, \mathrm{d}\sigma.$$

This requires some bookkeeping, but splitting this into the parts $(1/4, 1/3)$, $(1/3, 1/2)$ and $(1/2, c)$ one quickly verifies that no term contriibutes more that $x^{1+\varepsilon}$. To this end, we showed

$$\sum_{n \leq x \text{ sqfull}} 1 = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + O\left(\frac{x^{1+\varepsilon}}{T} + x^{1/4+\varepsilon} \right).$$

The claim follows upon setting $T = x^{3/4}$.

b - 5p) The good thing with smooth weights is that their mellin transforms usually decay quickly along vertical lines and we do not have to worry about cutting off the integral. Perron's formula reveals with $c > 1/2$

$$\sum_{n \text{ squarefull}} \mathrm{e}^{-n/x} = \frac{1}{2\pi \mathrm{i}} \int_{(c)} \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)} x^s \Gamma(s) \, \mathrm{d}s.$$

As Γ vanishes rapidly along vertical lines, we can shift the contour to $\operatorname{Re} s = 1/6 + \varepsilon$ and obtain

$$\cdots = \frac{1}{2} \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{1}{3} \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + \frac{1}{2\pi \mathrm{i}} \int_{(1/6+\varepsilon)} \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)} x^s \Gamma(s) \, \mathrm{d}s.$$

The integral is absolutely convergent, hence gives an error of size $O(x^{1/6+\varepsilon})$.

Remark: We will later prove that $\zeta(s)$ does not have zeroes in some neighbourhood of the line $\operatorname{Re} s = 1$, which in particular implies that there are no zeroes on the line itself. Hence we can get even shift the contour onto $\operatorname{Re} s = 1/6$, killing the $+\varepsilon$.

Problem 4

a - 6p) Every finite abelian group can be decomposed as a product of cyclic groups of prime-power-order. Hence the number of isomorphism classes of abelian groups of order n gives a multiplicative arithmetic function

$$a : \mathbb{N} \to \mathbb{N}, \quad n \mapsto \#(\{\text{abelian groups of order } n\}/\cong).$$

If $n = p^r$ is a prime power, we find that $a(n)$ is given by the number of (additive) partitions of r. Indeed, to a partition

$$1 \cdot a_1 + 2 \cdot a_2 + 3 \cdot a_3 + \cdots = r$$

we can associate a group $(\mathbb{Z}/p\mathbb{Z})^{a_1} \times (\mathbb{Z}/p^2\mathbb{Z})^{a_2} \times (\mathbb{Z}/p^3\mathbb{Z})^{a_3} \times \ldots$ of order p^r, and vice versa. One quickly verifies (at least formally), that

$$\sum_{n=1}^{\infty} a(n) x^n = (1 + x + x^2 + \ldots)(1 + x^2 + x^4 + \ldots)(1 + x^3 + x^6 + \ldots) \cdots$$

and substituting $x = p^{-s}$ for varying p yields the desired formula

$$\sum_{n=1}^{\infty} a(n) n^{-s} = \prod_p \prod_{r=1}^{\infty} (1 - p^{-rs})^{-1} = \prod_{r=1}^{\infty} \zeta(rs).$$

The last step might demand clearification. Remember that a product $\prod a_n$ with $a_n \neq 0$ converges absolutely to something $\neq 0$ iff the sum $\sum |a_n - 1|$ converges absolutely. In $\operatorname{Re} s > 1 + \delta$ we have the uniform bound

$$|1 - \zeta(rs)| \ll \sum_{n=2}^{\infty} n^{r(-1-\delta)} \ll_\delta 2^{-r},$$

so that which shows that indeed, the product converges absolutely and locally uniformly in $\operatorname{Re} s > 1$.

b - 4p) The heuristic goes as follows. Let F be the Dirichlet series attached to a. By the above, F is a holomorphic function for $s > 1$, but by the continuation of the first ζ-factor, we find that F has a continuation to a meromorphic function on $\mathrm{Re}\, s > 1/2$. (Aside: We can apply the functional equation to as many ζ-factors as we want, yielding continuations to $\mathrm{Re}\, s > 1/n$ for arbitrarily large $n \in \mathbb{N}$. But F can never be meromorphically continued to all of $\mathbb{C}$. This is because there are poles at $s = 1, 1/2, 1/3, \ldots$, which by the identity theorem implies that $F^{-1} = 0$.) Now Perron's Formula reads

$$\sum_{n \le x} a(n) = \frac{1}{2\pi \mathrm{i}} \int_{(c)} F(s) x^s \frac{\mathrm{d}s}{s},$$

and upon shifting the contour to $1 - \varepsilon$ we obtain

$$\sum_{n \le x} a(n) = x \mathrm{Res}_{s=1} F(s) + \frac{1}{2\pi \mathrm{i}} \int_{(1-\varepsilon)} F(s) x^s \frac{\mathrm{d}s}{s}.$$

The residue is given by $C = \zeta(2)\zeta(3)\cdots$, and we'd hope that we would be able to approximate the integral by something of size $o(x)$.

Proving the asymptotic. Proving the asymptotic is quite challenging, as we would have to find some bound on $a(n)$ to apply (4.7). The convergence of $\sum_n a(n) n^{-s}$ for $\mathrm{Re}(s) > 1$ gives $a(n) \ll n^{1+\varepsilon}$, but there is no trivial way to get anything beyond that. But it turns out we don't need such bounds! Note that we really need to include a bound of $a(n)$ in (4.7) because we try to approximate a function that "jumps" (the LHS) with a function that is continuous in x (the integral, at least as long as $T = T(x)$ is continuous in x). But if we decide to inspect the approximation away from the jumps of the LHS, we might be able to prove an error not involving terms of the form $O(\max_{n \sim x} |a_n|)$. This idea is sketched in the following.

Using a modified version of (4.7). The probably more sensible way to do this is to use a modified version of (4.7): If we assume $x \in \frac{1}{2} + \mathbb{N}$ (more generally, $x \in [\delta, 1 - \delta] + \mathbb{N}$ works for $0 < \delta < 1/2$), we can copy the proof of (4.7), but the first summand A_x can be avoided. This gives (with the same terminology as in (4.7)) the statement

$$\sum_{n \le x} a_n = \frac{1}{2\pi \mathrm{i}} \int_{c - \mathrm{i}T}^{c + \mathrm{i}T} \sum_{n \in \mathbb{N}} \frac{a_n}{n^s} x^s \frac{\mathrm{d}s}{s} + O\left(\frac{x^c}{T} \sum_n \frac{|a_n|}{n^c} + A_x \frac{x \log x}{T} \right).$$

We can now follow the same strategy as usual, and in the end realize that $T = x^\alpha$ can be chosen an arbitrary power of x, which should ultimately yield a asymptotic with error $O(x^{1/2+\varepsilon})$. (You will need $A_x = \max_{n \sim x} |a_n| \ll x^{1+\varepsilon}$.) This is left as an exercise :)

The following solution introduces a new idea. We sacrifice a bit of error size, but get a smooth ride when moving the integral to the left in exchange. You will realize we almost don't have to worry about messy calculations at all!

Proving the asymptotic using Cesàro-weights. Instead of trying to avoid the jumps, we could also try to smooth out the LHS of (4.7). Instead of bounding

$$S_0(x) = \sum_{n \le x} a(n),$$

we try to bound

$$S_1(x) = \sum_{n \le x} a(n)(x - n) = \int_1^x S_0(y)\, \mathrm{d}y.$$

(These weights are called Cesàro weights). We hope to recover information about S_0 afterwards. Integrating Perron's formula, we find that

$$S_1(x) = \frac{1}{2\pi \mathrm{i}} \int_{(c)} F(s) x^{s+1} \frac{\Gamma(s)}{\Gamma(s+2)} \,\mathrm{d}s.$$

The Γ-factor is essentially bounded by $|s|^{-2}$, at least for $|s| > 2$. Whenever $\sigma > 1/2 + \delta$ and $|t| > 1$ we find

$$F(\sigma + \mathrm{i}t) \ll |\zeta(s)| \, \zeta(1+2\delta)\zeta(3/2 + 3/2\delta) \cdots \ll |t|^{\frac{1-\sigma}{2}+\varepsilon} \delta^{-1}.$$

Hence we can shift the contour to $\operatorname{Re} s = 1/2 + \delta$, pick up a pole and the remaining integral remains absolutely convergent. In formulas,

$$S_1(x) = \frac{x^2}{2} C + \int_{(1/2+\delta)} F(s) x^{s+1} \frac{\Gamma(s)}{\Gamma(s+2)} \,\mathrm{d}s = \frac{x^2}{2} C + O_\delta(x^{3/2+\delta}).$$

Nice, this at least shows that there is an asymptotic *on average*. But how can we make use of this? We also showed that the Lindelöf-Hypothesis is true *on average*, but we are far from proving the Lindelöf-Hypothesis in general! What plays in our favor here is that S_0 is non-decreasing. Denote by $E_0(x)$ the error function $S_0(x) - Cx$, and define E_1 as the integral of E_0. Note that we have $E_1(x) \ll x^{3/2+\varepsilon}$ by the above. We also make a choice of some $Q = x^\alpha$ for $\alpha \in [0, 1]$ and get (using monotonicity of S_0)

$$E_1(x+Q) - E_1(x) = \int_x^{x+Q} E_0(t) \,\mathrm{d}t \geq Q(S_0(x) - Cx - CQ) = QE_0(x) + O(Q^2).$$

But we also know that $E_1(x+Q) - E_1(x) = O(x^{3/2+\varepsilon})$, implying

$$QE_0(x) \leq O(x^{3/2+\varepsilon} + Q^2).$$

This shows $E_0(x) \leq O(x^{3/4+\varepsilon})$ once we choose $Q = x^{3/4}$. A similar lower bound can be established by inspecting $\int_{x-Q}^x E_0(t)\,\mathrm{d}t$ (exercise, haha). This proves $S_0(x) = Cx + O(x^{3/4+\varepsilon})$. This really is remarkable, as this in particular implies that $a(n) \ll n^{3/4+\varepsilon}$, which is a bound we did not know existed beforehand. Even more, this followed only from a bound on the vertical growth of $F(s)$ and the fact that $a(n) \geq 0$. Also note that we by did not do as good as we could have! We could have shifted further to the left and picked up more residues.

Analytic Number Theory
Problem Set 8

Problem 1. a) Show directly that

$$e^{-x} = \frac{1}{2\pi i}\int_{(1)} \Gamma(s) x^{-s} ds$$

by shifting the contour to the left and picking up the residues. *Hint:* Use Problem 2 of Set 5.

b) Compute the polynomials P_2 and P_3 in Example (4.9) in terms of the Taylor coefficients of $\zeta(s)$ at $s = 1$. Compare with (1.6b) to compute the zero-th Taylor coefficient of $\zeta(s)$.

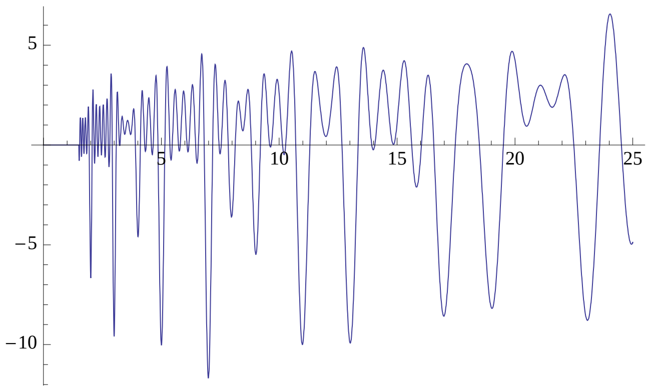
Problem 2. a) Let b be the constant in (5.2) for the Riemann zeta-function. Show that $\zeta'(0) = -\frac{1}{2}\log(2\pi)$ and conclude that $b = \frac{1}{2}(-\gamma - 2 + \log(4\pi)) \approx -0.023\ldots$.. *Hint:* Use (5.2) with $s = 0$ and $\Gamma'(1) = -\gamma$.

b) Conclude that $|\Im\rho| \geqslant 6$ for all zeros of the Riemann zeta function. *Hint:* Use (5.3a)

Problem 3. a) Give a rigorous proof of Theorem 5.4.

b) Let $\rho_1 = 1/2 + i\gamma_1, \rho_2 = 1/2 + i\gamma_2, \ldots$ denote the (non-trivial) zeros of the Riemann zeta function with $0 < \gamma_1 < \gamma_2 \ldots$. Here is a picture of

$$\sum_{j=1}^{30} \cos(\beta_j \log x). \tag{1}$$



with $\gamma_{30} = 101.318\ldots$ What do see? Explain!

Hint: No rigorous proof is required here, just a qualitative answer. Interpret the sum (1) as the main term of the right hand side of the explicit formula with a suitable function $\widehat{w}(s)$. An interesting choice is the function

$$w(y) = w_{S,x}(y) = \frac{S}{2(\pi y)^{1/2}} \exp\left(-\left(\frac{S}{2}\log\left(\frac{y}{x}\right)\right)^2\right).$$

Verify that $\widehat{w}(s) = x^{s-1/2} \exp\left(\left(\frac{s-1/2}{S}\right)^2\right)$, so $\widehat{w}(1/2+i\gamma) = x^{i\gamma}\exp(-(\gamma/S)^2)$. Choose $S \approx 100$, and convince yourself that for $x \leqslant 25$, w is essentially a large peak at $y = x$.

Problem 4. [exercise in partial summation] Conclude from (5.6) that

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O\left(x\exp(-c\sqrt{\log x})\right)$$

for some constant $c > 0$.

Due: Tue, Dec 6

Solutions to Sheet 8.

Problem 1

a-4p) Look at sheet 5.

b-6p) The approximation in (4.9) reads

$$\sum_{n\leq x} d_k(n) = xP_k(\log x) + O(x^{1-\delta}).$$

Reading the proof reveals that the main term is given by the residue

$$R := \mathrm{Res}_{s=1}\frac{\zeta^k(s)x^s}{s} = \mathrm{Res}_{s=1}F_k(s),$$

where for convenience $F_k(s) := \frac{\zeta^k(s)x^s}{s}$. Of course $\zeta^k(s)$ has a singularity of degree k at 1, and we only need to calculate the (-1)st term of the Laurent expansion of F at 1. We have the Taylor expansions

$$\frac{1}{s} = \sum_{n=0}^{\infty}(-1)^n(s-1)^n = 1-(s-1)+(s-1)^2+O((s-1)^3)$$

and

$$x^s = \sum_{n=0}^{\infty}\frac{x(\log x)^n}{n!}(s-1)^n = x + x(\log x)(s-1) + \frac{1}{2}x(\log x)^2(s-1)^2 + O((s-1)^3).$$

Let a_n denote the coefficients of the Laurent series of ζ at 1, i.e.

$$\zeta(s) = \sum_{n=-1}^{\infty} a_n(s-1)^n.$$

Calculating P_2 and P_3 now is pure calculation.

Calculating P_2. We find

$$\zeta^2(s) = \left(\sum_{n=-1}^{\infty} a_n s^n\right)^2 = a_{-1}^2(s-1)^{-2} + 2a_{-1}a_0(s-1)^{-1} + O(1).$$

We multiply this with the Taylor series above and find that the coefficient of $(s-1)^{-1}$ is given by

$$2a_{-1}a_0x + a_{-1}^2(x\log x - x) = x(a_{-1}^2\log x + 2a_{-1}a_0 - a_{-1}^2).$$

Remark. It is possible to show by elementary means that

$$\sum_{n\leq x}^{\infty} d_2(n) = x\log x + (2\gamma-1)x + O(x^{1/2}),$$

which shows that $a_0 = \gamma$. (I just realized that $a_{-1} = 1$ is already known haha, but we could probably also derive this with a similar approach and $k = 1$). This shows

$$P_2(X) = X + 2a_0 - 1.$$

Calculating P_3. We find similarly to above

$$\zeta^3(s) = (s-1)^{-3} + 3a_0(s-1)^{-2} + 3(a_0^2 + a_1)(s-1)^{-1} + O(1).$$

and again use this to figure out the coeffitient of $(s-1)^{-1}$ in the Laurent expansion of F_3 around $s = 1$. We find that this coefficient is given by

$$\begin{aligned} &3(a_0^2 + a_1)x + 3a_0x(\log x - 1) + x((\log x)^2 - \log x + 1) \\ &\qquad = x(\frac{1}{2}(\log x)^2 + (\log x)(3\gamma - 1) + 3(\gamma^2 + a_1 - \gamma) + 1), \end{aligned}$$

i.e.

$$P_3(X) = \frac{1}{2}x^2 + (3\gamma - 1)x + 3(a_1 + a_0^2 - a_0) + 1.$$

Problem 2

a) To calculate $\zeta'(0)$, we make use of the functional equation, in the form

$$\zeta(1-s) = \zeta(s) \cdot \frac{2\Gamma(s)}{(2\pi)^s} \sin((\pi(1-s)/2).$$

(This can be derived from the usual functional equation using the reflection formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$. For s close to 1 we find

$$(s-1)\zeta(s)\Gamma(s) = 1 + O((s-1)^2),$$

and

$$\sin(\pi(1-s)/2) = -\tfrac{\pi}{2}(s-1) + O((s-1)^3).$$

Hence

$$\zeta(1-s) = -\frac{\pi}{(2\pi)^s} + O((s-1)^2) = -\frac{1}{2} - \frac{1}{2}\log(2\pi)(s-1) + O((s-1)^2).$$

This gives $\zeta'(0) = -\frac{\log 2\pi}{2}$. We now insert this into (5.2). For $L(s) = \zeta(s)$, this reads as

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{\log \pi}{2} + \frac{1}{2}\frac{\Gamma'(s/2)}{\Gamma(s/2)} - b + \frac{1}{s} + \frac{1}{s-1} - \sum_{\rho \neq 0,1} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

Taking the logarithmic derivative of the recurrece relation $\Gamma(s+1) = s\Gamma(s)$ reveals

$$\frac{\Gamma'(s+1)}{\Gamma(s+1)} = \frac{1}{s} + \frac{\Gamma'(s)}{\Gamma(s)}.$$

We can use this to simplify our equation, leaving us with

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{\log \pi}{2} + \frac{1}{2}\frac{\Gamma'(s/2+1)}{\Gamma(s/2+1)} - b + \frac{1}{s-1} - \sum_{\rho \neq 0,1} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

When inserting $s = 0$, this sum vanishes, and we find

$$-\frac{\zeta'(0)}{\zeta(0)} = -\frac{\log \pi}{2} + \frac{1}{2}\frac{\Gamma'(1)}{\Gamma(1)} - b - 1.$$

This proves the claim, as we know $\zeta(0) = -\frac{1}{2}$, $\Gamma(1) = 1$ and the values for ζ' and Γ' from above.

Remark. The formula $\Gamma'(1) = -\gamma$ can be derived from the Weierstraß product

$$\frac{1}{\Gamma(z)} = z\mathrm{e}^{\gamma z} \prod_{j=1}^{\infty} \left(1 + \frac{z}{j}\right) \mathrm{e}^{-z/j}$$

by taking logarithmic derivatives on both sides and inserting $z = 1$.

b) This statement is false, but of course, we are supposed to show $|\mathrm{Im}(\rho)| \geq 6$ for all the *non-trivial* roots of the zeta function. By (5.3a), we have that

$$-b = -\operatorname{Re} b = \sum_{\rho} \operatorname{Re}(\tfrac{1}{\rho}).$$

The idea is that if $\operatorname{Im} \rho_1$ was small, then this sum would be so large that this equality cannot hold (remember that $b = -0.023$ is quite small). Let $\rho = \sigma + it$ be a root with smallest possible imaginary value. Note that with ρ, we also have roots $1 - \sigma \pm it$ and $\sigma - it$, so we may assume that $\sigma \geq \frac{1}{2}$ and that $t > 0$. As all contributions in the sum of (5.3a) are positive, we find

$$-b \geq \frac{1}{\sigma + \mathrm{i}t} + \frac{1}{\sigma - \mathrm{i}t} = \frac{2\sigma}{\sigma^2 + t^2} \geq \frac{1}{1 + q^2}.$$

This shows $t \geq \sqrt{-b^{-1} - 1} \approx 6.5036$.

Problem 3

a-5p) The "only" thing left to made precise is the contour shift. The main ingrediants are (5.3b) and (5.3c).

Just for convenience, let's summarize the bounds we need for $\frac{\zeta'}{\zeta}$:

$$\frac{\zeta'}{\zeta}(s) = \begin{cases} O(1) & \operatorname{Re} s \geq 2, \\ O(\log|s|) & \operatorname{Re} s \leq -\frac{1}{2},\ |s + 2m| \geq \frac{1}{4} \text{ for all } m \in \mathbb{N}, \\ \sum_{|\rho - s| \leq 1} \frac{1}{s - \rho} + O(1 + \log|s|), & -1 \leq \operatorname{Re} s \leq 3. \end{cases}$$

The first bound follows from $\frac{\zeta'}{\zeta}(s) = \sum_n \Lambda(n) n^{-s}$, the second part follows from the first bound, the functional equation and Stirling's formula. The third part is (5.3c).

By (5.3b), there are approximately $\log T$ roots of the zeta-function with imaginary part close to T (i.e., $|T - \operatorname{Im} \rho| \leq 1$). Hence given some $n \in \mathbb{Z}$, the pigeonhole principle assures that it is possible to find some $T = T_n$ with $|n - T| \leq 1$ and $\min_\rho |T - \operatorname{Im} \rho| \leq \frac{1}{\log|n|}$. Together with (5.3$c$), this gives that $\frac{\zeta'}{\zeta}(\sigma + \mathrm{i}T) \ll (\log|T|)^2$ on that line.

The plan is now to choose some large n, and shifting the truncated integral

$$-\frac{1}{2\pi\mathrm{i}} \int_{2-\mathrm{i}n}^{2+\mathrm{i}n} \frac{\zeta'}{\zeta}(s) \widehat{\omega}(s)\, \mathrm{d}s \tag{1}$$

to $\operatorname{Re} s = -1/2$. This leaves us with the exercise to bound the horizontal integrals along the segments $[2 \pm \mathrm{i}n, -1/2 \pm \mathrm{i}n]$. By the rapid decay of $\widehat{\omega}$ and the bounds for ζ'/ζ, changing the boundaries of the integral in (1) from $[2 - \mathrm{i}n, 2 + \mathrm{i}n]$ to $[2 - \mathrm{i}T_n, 2 + \mathrm{i}T_n]$ comes only with a small cost of $o(1)$. So we may also assume that uniformly $\zeta(\sigma + \mathrm{i}T_n) \ll (\log n)^2$ along the horizontal segments. This justifies the first contour shift, and we obtain

$$\sum_n \Lambda(n)\omega(n) = \frac{1}{2\pi\mathrm{i}} \int_{2-\mathrm{i}n}^{2+\mathrm{i}n} \frac{\zeta'(s)}{\zeta(s)} \widehat{\omega}(s)\,\mathrm{d}s + o(1)$$

$$= \sum_{|\operatorname{Im}\rho| \le T_n} \widehat{\omega}(\rho) + \frac{1}{2\pi\mathrm{i}} \int_{-1/2-\mathrm{i}T_n}^{-1/2+\mathrm{i}T_n} \frac{\zeta'(s)}{\zeta(s)} \widehat{\omega}(s)\,\mathrm{d}s + o(1).$$

We may let $n \to \infty$, obtaining

$$\sum_n \Lambda(n)\omega(n) - \sum_\rho \widehat{\omega}(\rho) = \int_{(-1/2)} \frac{\zeta'(s)}{\zeta(s)} \widehat{\omega}(s)\,\mathrm{d}s.$$

In the proof of (4.4c) we can abuse the fact that $\operatorname{Supp}(\omega) \subset [2, \infty)$ to show that $\widehat{\omega}(s) \ll \frac{2^{-s}}{|\operatorname{Im}(s)|^N}$ for all N. This shows the bound

$$\int_{(-A+1/2)} \frac{\zeta'(s)}{\zeta(s)} \widehat{\omega}(s)\,\mathrm{d}s \ll 2^{-A},$$

justifying the shift $\operatorname{Re} s \to -\infty$.

b-5p) The observation is that this function has large negative peaks at the primes , and when $n = p^k$ is a prime power (to be fair, withouth knowing the explicit formula, this would be hard to guess). Although the solution will (implicitely) assume the Riemann conjecture, this illustrates the fact that ζ *knows everything about the primes.*

Okay, let's analyze what's happening here. First, we have

$$\cos(\gamma_j \log x) = \operatorname{Re}(\mathrm{e}^{\mathrm{i}\gamma_j \log x}) = \operatorname{Re}(x^{\mathrm{i}\gamma_j}),$$

so we are plotting the real part of the sum of $x^{\mathrm{i}\gamma}$ over the first few zeroes. If we want to interpret this as a sum $\sum_\rho \widehat{\omega}(\rho)$, we would like to choose ω in way such that $\widehat{\omega}(1/2 + \mathrm{i}\gamma) \approx x^{\mathrm{i}\gamma}$ for the zeroes we want to consider, and $\widehat{\omega}$ decaying rapidly after that range (ignoring the contribution of the trivial zeroes).

Apparently (not clear to me how to come up with this but hey it works) a convenient seems to be

$$\widehat{\omega}(1/2 + \mathrm{i}\gamma) = x^{\mathrm{i}\gamma} \exp(-(\gamma/S)^2) = x^{s-\frac{1}{2}} \exp\left(\left(\frac{s - 1/2}{S}\right)^2\right),$$

as this vanishes quickly once $\gamma > S$. We choose S to be a parameter roughly of the size of the largest zero we want to consider, which in our case is $\gamma_{30} \approx 100$. That's why we choose $S = 100$. We will later show that the weight

$$\omega(y) = \omega_{S,x}(y) = \frac{S}{2(\pi y)^{1/2}} \exp\left(-\left(\frac{S}{2}\log\left(\frac{y}{x}\right)\right)^2\right)$$

is the inverse Mellin-transform of $\widehat{\omega}$. Now this is large if the part in the exponential vanishes, i.e., if $x \approx y$. On the other side, if y is not close to x then $\log(y/x)$ becomes large (say of size $\approx \frac{10}{S}$), then the factor $\exp(-S^2/4\log(y/x)^2)$ makes ω decay quickly. So at least for x not too large (for large x we need a further distance between y and x to make $\log(y/x)$ become large), ω essentially looks like a peak at $y = x$. We are now ready to explain what's going on. With the explicit formula (which we are technically not even allowed to use as ω is not compactly supported, but whatever), we find

$$\sum_n \Lambda(n)\omega_{S,x}(n) \approx -\sum_{|\operatorname{Im}\rho|\le S} \widehat{\omega}(\rho) \approx \sum_{j=1}^{S} \cos(\gamma_j \log x).$$

If now $x \approx p^k$ is close to a prime power, the LHS is $\approx \Lambda(n)\frac{S}{y^{1/2}}$, large. If not, there is no term on the LHS that contributes much, so we would expect the RHS to be small.

Prove that "$\widehat{\omega} = \widehat{\omega}$". We put $\widehat{\omega}$ in the inverse mellin transform to find

$$\omega(y) = \frac{1}{2\pi\mathrm{i}} \int_{(c)} x^{s-1/2} \exp\left(\left(\frac{s-1/2}{S}\right)^2\right) y^{-s}\,\mathrm{d}s$$

for all real numbers c. We substitute $u = \frac{s-1/2}{S}$ and find

$$\omega(y) = \frac{S}{y^{1/2}} \cdot \frac{1}{2\pi\mathrm{i}} \int_{(c)} \exp(u^2)\left(\frac{y}{x}\right)^{-Su} \mathrm{d}u.$$

Abbreviating $v = S\log\frac{y}{x}$ shows further that

$$\frac{S}{y^{1/2}}\frac{1}{2\pi\mathrm{i}} \int_{(c)} \exp(u^2)\exp(-uv)\,\mathrm{d}u = \frac{S\exp(-v^2/4)}{y^{1/2}} \cdot \frac{1}{2\pi\mathrm{i}} \int_{(c)} \exp((u-v/2)^2)\,\mathrm{d}u.$$

This integral does not depend on c, hence we may wlog assume $c = v/2$, which reveals that this integral equals

$$\frac{1}{2\pi\mathrm{i}} \int_{(0)} \exp(u^2)\,\mathrm{d}u = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-t^2)\,\mathrm{d}t = \frac{1}{2\sqrt{\pi}},$$

just what we wanted.

Problem 4

First, an aside on the weird-looking error term $\psi(x) - x \ll x\mathrm{e}^{-c\sqrt{\log x}}$. On the one side it is better than every error term of the form $x/(\log x)^A$ (for $A \in \mathbb{R}_{>0}$ large), on the other side it is worse than every error term of the form $x^{1-\delta}$ would be (for $\delta \in \mathbb{R}_{>0}$ small).

Our version of the prime number theorem reads

$$\psi(x) = \sum_{p^n \le x} \log p = x + O(x\mathrm{e}^{-c\sqrt{\log x}})$$

for some constant $c > 0$. We deduce a formula for π in two steps. First we show that $\psi(x)$ does not differ too much from the weighted prime-counting function

$$\psi_0(x) := \sum_{p \le x} \log p.$$

Then we use ψ_0 for partial summation, utilizing that

$$\pi(x) = \sum_{p\le x} \frac{\log p}{\log p} = \frac{\psi_0(x)}{\log x} + \int_2^x \frac{\psi_0(t)}{t(\log t)^2}\,\mathrm{d}t. \tag{2}$$

Evaluating this should be possible using the approximation for $\psi_0(x)$.

Let's carry this through, beginning with the estimate for $|\psi(x)-\psi_0(x)|$. We find

$$\psi(x)-\psi_0(x)=\sum_{p^k\le x,\ k\ge 2}\log p\le\left(\sum_{p\le\sqrt{x}}+\sum_{p\le x^{1/3}}+\cdots+\right)\log x$$

Note that there are at most $\log_2 x$ summation signs which don't run over an empty set, and every index set contains (trivially) less than $\sqrt{x}$ primes. We obtain

$$\psi(x)-\psi_0(x)\le(\log_2 x)\sqrt{x}(\log x)\ll x^{1/2+\varepsilon}.$$

Now ψ_0 satisfies the same approximation as ψ, as

$$\psi_0(x)=\psi(x)+O(x^{1/2+\varepsilon})=x+O(x\mathrm{e}^{-c\sqrt{\log x}}).$$

Inserting this in (1) yields

$$\pi(x)=\frac{x}{\log x}+\int_2^x\frac{1}{(\log t)^2}\,\mathrm{d}t+O(x\mathrm{e}^{-c\sqrt{\log x}}),$$

where we used that $\int_2^x\frac{1}{t(\log t)^2}\,\mathrm{d}t\ll 1$. As

$$\int_2^x\frac{1}{(\log t)^2}\,\mathrm{d}t=\left[\mathrm{Li}(t)-\frac{t}{\log t}\right]_2^x=\mathrm{Li}(x)-\frac{x}{\log x}+O(1),$$

the claim follows.

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Analytic Number Theory
Problem Set 9

Problem 1.-2. Show that

$$\sum_{\substack{n \leqslant x \\ \Omega(n)=2}} 1 \sim \frac{x \log\log x}{\log x}$$

(in other words, the quotient of the two sides tends to 1 as x tends to infinity).
Hint: Careful partial summation and the prime number theorem. Little exercise in calculus: what is $\frac{d}{dp}\int_p^{x/p} \frac{dt}{\log t}$?
Remarks: Heuristically, the result is easy to guess. One has

$$\sum_{pq \leqslant x} 1 = \sum_{p \leqslant x} \sum_{q \leqslant x/p} 1 \approx \sum_{p \leqslant x} \frac{x/p}{\log x/p} \approx \sum_{p \leqslant x} \frac{x/p}{\log x} \approx \frac{x \log\log x}{\log x}.$$

The "real" proof is more technical, of course (start by distinguishing $p \leqslant q$ and $p \geqslant q$). We will show later that

$$\sum_{\substack{n \leqslant x \\ \Omega(n)=k}} 1 \sim \frac{x(\log\log x)^{k-1}}{(k-1)!\log x}$$

for $k \in \mathbb{N}$. This makes perfect sense, since (formally!) summing over k gives

$$x = \sum_{n \leqslant x} 1 = \sum_k \sum_{\substack{n \leqslant x \\ \Omega(n)=k}} 1 \approx \sum_k \frac{x(\log\log x)^{k-1}}{(k-1)!\log x} = \frac{x}{\log x}\exp(\log\log x) = x.$$

Problem 3. Show $\phi(n) \gg n/\log\log n$ for $n \geqslant 3$.

Problem 4. Let $c > 0$. Let $q_1 < q_2 < \dots$ be the sequence of "exceptional" moduli for which there exists a primitive character χ mod q such that $L(s,\chi)$ has a real zero $\beta > 1 - c/\log q$ (Siegel zero). Show that for c sufficiently small, one has $q_{j+1} > q_j^2$. Conclude that there are at most $O(\log\log X)$ exceptional moduli up to X.
Hint: This follows quickly from (5.10).

Due: Tue, Dec 13

Solutions to Sheet 9.

Problem 1&2

Want to estimate

$$S_2(x) := \sum_{n\le x, \Omega(n)=2} 1.$$

Write it as

$$S_2(x) = \sum_{p\le\sqrt{x}} \sum_{p\le q\le x/p} 1 = \sum_{p\le\sqrt{x}} (\pi(x/p)-\pi(p)) + O(\sqrt{x}).$$

Use PNT, get

$$S_2(x) = \sum_{p\le\sqrt{x}} \int_p^{x/p} \frac{\mathrm{d}t}{\log t} + O(x\mathrm{e}^{-c\sqrt{\log x}}).$$

for a constant $c>0$ (not the same as in the PNT). Concept-wise we are done here, as all is left to do is to do partial summation with $g(t) = \mathrm{Li}(x/t) - \mathrm{Li}(t)$ as smooth weight, and a_n the indicator function on primes. Estimating the rest is a bit tedious, but straight-forward:

We have $g(\sqrt{x}) = 0$ and $-g'(t) = \frac{1}{\log t} + \frac{x}{t^2\log(x/t)}$. We obtain

$$S_2(x) = \sum_{p\le\sqrt{x}} g(p) = \int_2^{\sqrt{x}} \frac{\pi(t)}{\log t} + \frac{\pi(t)x}{t^2\log(x/t)}\,\mathrm{d}t.$$

The integral over $\pi(t)/\log t$ can be dealt with quite quickly. We have $\pi(t) \ll \frac{t}{\log t}$, hence

$$\int_2^{\sqrt{x}} \frac{\pi(t)}{\log t}\,\mathrm{d}t \ll \int_2^{\sqrt{x}} \frac{t}{(\log t)^2}\,\mathrm{d}t \ll \frac{x}{(\log x)^2}.$$

We are left with

$$S_2(x) = \int_2^{\sqrt{x}} \frac{\pi(t)x\,\mathrm{d}t}{t^2\log(x/t)} + O\left(\frac{x}{(\log x)^2}\right) = \int_2^{\sqrt{x}} \frac{x((\log t)^{-1} + O((\log t)^{-2}))}{t\log(x/t)}\,\mathrm{d}t + O\left(\frac{x}{(\log x)^2}\right),$$

where we applied the PNT again, this time with error term $O(x/(\log x)^2)$. The integral over the O-term is also easily handled. We have $\log(x/t) \gg \log x$, and hence find that the contribution is bounded by

$$\frac{x}{\log x}\int_2^{\sqrt{x}} \frac{1}{t(\log t)^2}\,\mathrm{d}t \ll \frac{x}{\log x}.$$

We are left with

$$S_2(x) = x\int_2^{\sqrt{x}} \frac{1}{t(\log t)(\log\frac{x}{t})}\,\mathrm{d}t + O(\frac{x}{\log x}).$$

We can use the geometric series to show that

$$\frac{1}{\log\frac{x}{t}} = \frac{1}{\log x(1-\frac{\log t}{\log x})} = \frac{1}{\log x}\left(1+O(\tfrac{\log t}{\log x})\right) = \frac{1}{\log x} + O\left(\frac{\log t}{(\log x)^2}\right).$$

Hence we obtain

$$S_2(x) = \frac{x}{\log x}\int_2^{\sqrt{x}} \frac{1}{t\log t}\,\mathrm{d}t + O\left(\frac{x}{(\log x)^2}\int_2^{\sqrt{x}}\frac{1}{t}\right) + O(\tfrac{x}{\log x}) = \frac{x}{\log x}\int_2^{\sqrt{x}} \frac{1}{t\log t}\,\mathrm{d}t + O\left(\frac{x}{\log x}\right).$$

This integral is exactly given by

$$\int_2^{\sqrt{x}} \frac{1}{t\log t}\,\mathrm{d}t = \log\log\sqrt{x} - \log\log 2,$$

which leaves us with

$$S_2(x) = \frac{x\log\log x}{\log x} + O\left(\frac{x}{\log x}\right),$$

as desired.

Problem 3

This is a consequence of Merten's theorem, which states that for $x > 1$,

$$\sum_{p\le x}\frac{1}{p} = \log\log x + C + O((\log x)^{-1})$$

for some constant C.

Note that

$$\frac{\varphi(n)}{n} = \prod_{p\mid n}(1-p^{-1}),$$

so we really want to show that the RHS is $\gg (\log\log n)^{-1}$. The product over the prime divisors of n is hard to get a hold on. It would be much easier if we could somehow relate this to products of the form $\prod_{p\le x}(1-p^{-1})$, as these products can be bounded with Merten's formula:

$$\begin{aligned}\prod_{p\le x}\left(1-\frac{1}{p}\right) &= \exp\left(\sum_{p\le x}\log\left(1-\frac{1}{p}\right)\right) = \exp\left(-\sum_{p\le x}\frac{1}{p} - \sum_{p\le x}\sum_{k\ge 2}\frac{1}{kp^k}\right)\\ &= \exp\left(-\log\log x - C + O((\log x)^{-1}) - \sum_{p}\sum_{k\ge 2}\frac{1}{kp^k} + O\left(\sum_{p>x}\sum_{k\ge 2}\frac{1}{p^k}\right)\right)\\ &= \frac{\mathrm{e}^{-C'}}{\log x}\exp\left(O\left(\frac{1}{\log x}\right)\right) = \frac{\mathrm{e}^{-C'}}{\log x}(1+O((\log x)^{-1}) \gg \frac{1}{\log x}.\end{aligned}$$

(This also was on sheet 0). In particular, if we choose $x = \log n$, we obtain

$$\prod_{p\le\log n}\left(1-\frac{1}{p}\right) \gg (\log\log n)^{-1}.$$

This is nice, because the prime divisors $p \mid n$ with $p \ge \log n$ don't contribute anything:

$$\prod_{p\mid n,\ p\le\log n}\left(1-\frac{1}{p}\right) \ge \left(1-\frac{1}{\log n}\right)^{\omega(n)} \ge \left(1-\frac{1}{\log n}\right)^{2\log n} \gg 1.$$

(Here we used $\omega(n) \le \log_2(n) \le 2\log n$ and that one formula for e). Hence we can conclude

$$\frac{\varphi(n)}{n} \ge \left(1-\frac{1}{\log n}\right)^{\omega(n)}\prod_{p\le\log n}\left(1-\frac{1}{p}\right) \gg \frac{1}{\log\log n}.$$

Notes after correcting. I just realized that the long calculation can be replaced by a reference to (5.9). This also makes the reference to Merten's theorem dispensable, but technically uses the (much stronger) prime number theorem.

Problem 4

Okay, let $c > 0$ and let q and q' be two exceptional moduli with zeroes characters χ, χ' and real zeroes β, β' satisfying the condition of the exercise. Let's compare the assumptions with the statement of (5.12).

(A) We have $1 - \frac{c}{\log q} < \beta$, and similar for q'.

(5.12) There is some small $d > 0$ (independent of q and q') such that we have $\min(\beta, \beta') \leq 1 - \frac{d}{\log(qq')}$.

If we assume $q < q'$, we certainly obtain

$$1 - \frac{c}{\log q} < 1 - \frac{d}{\log(qq')}, \quad \text{i.e.} \quad \frac{d}{c} < \frac{\log(qq')}{\log q}, \quad \text{i.e.} \quad q' > q^{d/c-1}.$$

Thus, any $c < d/3$ does the job.

This shows that there are $O(\log\log n)$ exceptional moduli up to n.

Aside: There is nothing special about the 2 in the exponent, if we choose c small enough we can get arbtirarily large exponents. But gives stronger conditions on what it means to be exceptional.

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Prof. Dr. Valentin Blomer winter term 2022/23

Analytic Number Theory
Problem Set 10

Problem 1. Let $(a, q) = 1$. Find an upper bound for the smallest prime $p \equiv a \,(\text{mod } q)$ as a function of q. Show that under the assumption of the Generalized Riemann Hypothesis for Dirichlet L-functions there exists a prime $p \equiv a \pmod q$ such that $p \ll q^{2+\varepsilon}$.

Problem 2. Let χ be a non-trivial character modulo q.
a) Let $M \leqslant N$ and $\Re s > 0$. Show that

$$\sum_{M < n \leqslant N} \chi(n) n^{-s} \ll \frac{q|s|M^{-\Re s}}{\Re s}.$$

b) Show $L(s, \chi) \ll \log q$ for $\Re s > 1 - 1/\log q$, $|\Im s| \leqslant q$.
c) Show $L'(s, \chi) \ll (\log q)^2$ for $\Re s > 1 - 1/\log q$, $|\Im s| \leqslant q$. *Hint:* you can either mimic the proof of b) or apply Cauchy's integral formula.

Problem 3. Fix any $A > 0$. Let $R(n)$ denote the number of ways of writing $n \in \mathbb{N}$ as a sum of a prime and a square-free number. Show the asymptotic formula

$$R(n) = \prod_{p \nmid n} \left(1 - \frac{1}{p(p-1)}\right) \int_2^n \frac{dt}{\log t} + O\left(\frac{n}{(\log n)^A}\right).$$

Hint: Start with $R(n) = \sum_{p<n} \mu(n-p)^2$ and establish the convolution formula $\mu^2(n) = \sum_{d^2 | n} \mu(d)$. Use Siegel-Walfisz where applicable and estimate the rest trivially.

Problem 4. Show that for almost all numbers $n \leqslant x$ there exist natural numbers a, b with $4/n = 1/a + 1/b$. Here "almost all" means that the set of numbers not having this property has cardinality $o(x)$.
Hint: Show that each $n \equiv 3 \,(\text{mod } 4)$ has this property and conclude that all numbers n having a prime divisor $p \equiv 3 \,(\text{mod } 4)$ have this property. Now use (5.15).
Remark: The famous unsolved Erdős-Straus conjecture states that for *all* numbers n there exist a, b, c such that $4/n = 1/a + 1/b + 1/c$.

Due: Tue, Dec 20

Zoom login for the lecture on Dec 23:

https://uni-bonn.zoom.us/j/64362904429?pwd=S1FoL21rS2VYS2hodkc4NHZpa1Jndz09
Meeting ID: 643 6290 4429
Passcode: 555095

Solutions to Sheet 10.

Reminder: $\mathrm{Li}(n) \coloneqq \int_2^n \frac{1}{\log t}\,\mathrm{d}t$.

Problem 1

This exercise tests your understanding of the Siegel-Walfiz theorem. Let's write down explicitely what it says.

Theorem 1 (Explicit Siegel-Walfisz)**.** *Let $A > 0$. There is a constant $K = K(A)$ and a constant c such that whenever $q < (\log x)^A$, we have the approximation (with K and c independent of q!!!)*

$$\left|\frac{x}{\varphi(q)} - \psi(x; q, a)\right| < Kx\mathrm{e}^{-c\sqrt{\log x}}.$$

It is a routine exercise in partial summation to obtain the corresponding statement for $\pi(x)$, which reads (with the same c)

Theorem 2 (Explicit Siegel-Walfisz for π)**.** *Let $A > 0$. There is a constant $K = K(A)$ and a constant c such that whenever $q < (\log x)^A$, we have the approximation (with K and c independent of q!!!)*

$$\left|\frac{\mathrm{Li}(x)}{\varphi(q)} - \pi(x; q, a)\right| < Kx\mathrm{e}^{-c\sqrt{\log x}}.$$

In particular, if q is large enough and we choose x such that $q < \log(x)^A$ (i.e., so large that we can apply Siegel-Walfisz), we have $Kx\mathrm{e}^{-c\sqrt{\log x}} < \frac{\mathrm{Li}(x)}{\varphi(q)} + 1$, so that $\pi(x; q, a) > 0$. The condition $q < (\log x)^A$ is equivalent to $\mathrm{e}^{q^{1/A}} < x$. As A may be chosen arbitrarily large, this implies that we have $\pi(x; q, a) > 0$ if $x \gg \mathrm{e}^{q^{\varepsilon}}$.

This bound might feel unsatisfying, because $\exp(q^{\varepsilon})$ is huge compared to q! We cannot do much better because the possibility of Siegel-Zeroes forces us to impose hard restrictions on the size of q compared to x. However, if the generalized Riemann hypothesis were true, we wouldn't have to worry about them. Perron's formula would the estimate

$$\psi(x, \chi) \ll (\log q) x^{\frac{1}{2}+\varepsilon}$$

and hence

$$\psi(x; q, a) = \frac{1}{\varphi(q)} \sum_{\chi} \sum_{n} \chi(n)\Lambda(n) n^{-s} = \frac{x}{\varphi(q)} + O((\log q) x^{1/2+\varepsilon}). \tag{1}$$

(I am not completely sure with the error term, but you might be able to work this out yourself. You will need the approximations

$$\frac{L'}{L}(s, \chi) = \begin{cases} O(1) & \mathrm{Re}\, s \geq 2 \\ O(\log q\,|s|) & \mathrm{Re}\, s \leq -\frac{1}{2} \text{ and } |s + m| > \frac{1}{4} \forall m \in \mathbb{N} \\ \sum_{|t - \mathrm{Im}\,\rho| \leq 1} \frac{1}{s-\rho} + O(\log(q(2 + |t|))) & -\frac{1}{2} \leq \mathrm{Re}\, s \leq 2, \end{cases}$$

where the latter sums goes over the non-trivial zeroes of $L(s, \chi)$.) Anyways, we observe that the main term of (1) dominates the error if $q^{2+\varepsilon} < x$. This is the desired bound.

Problem 2

(a) Let's try partial summation in conjunction with Polya-Vinogradov.

$$\sum_{M<n\le N}\chi(n)n^{-s} = N^{-s}\sum_{n\le N}\chi(n) - M^{-s}\sum_{n\le M}\chi(n) + s\int_M^N t^{-s-1}\sum_{M<n\le t}\chi(n)\,\mathrm{d}t$$

Now Polya-Vinogradov gives that every sum can be bound by $O(q^{1/2}\log q)$. We obtain

$$\sum_{M<n\le N}\chi(n)n^{-s} \ll M^{-\operatorname{Re}s}q^{\frac{1}{2}}\log q + |s|\int_M^N t^{-\operatorname{Re}s-1}q^{\frac{1}{2}}\log q\,\mathrm{d}t \ll \frac{|s|\,qM^{-\operatorname{Re}s}}{\operatorname{Re}s}.$$

Here we completed the integral and bounded $q^{\frac{1}{2}}\log q \ll q$. (This is not optimal, but it doesn't matter).

(b) Note that in part a, we can choose N arbitrarily large (without changing the implicit constant in $\ll$!). Hence it makes sense to choose some $M > 2$ and split the sum $L(s,\chi) = \sum_{n\in\mathbb{N}}\chi(n)n^{-s}$ into the parts $n \le M$ and $n > M$ and apply the result of part a for the latter sum. How large do we have to choose M in order to make this work? As $\operatorname{Re}s > 1-(\log q)^{-1}$ and $|\operatorname{Im}s| < q$ we find $|s| \ll q\operatorname{Re}s$. With part a, this gives

$$\sum_{M<n}\chi(n)n^{-s} \ll q^2M^{(\log q)^{-1}-1}.$$

If we choose $M = q^2$, this reduces to $\ll 1$, so let's see if the sum with terms $n < M$ is small enough. We trivially bound

$$\sum_{n<M}\chi(n)n^{-s} \ll \sum_{n<M} n^{(\log q)^{-1}n^{-1}} \ll \int_1^M t^{(\log q)^{-1}-1}\,\mathrm{d}t = \Big[(\log q)t^{(\log q)^{-1}}\Big]_1^M.$$

As $M = q^2$ and $(q^2)^{(\log q)^{-1}} = \mathrm{e}^{2(\log q)(\log q)^{-1}} = \mathrm{e}^2 \ll 1$, we are done.

(c) We will prove this with Cauchy's integral formula. Remember what it says:

$$L'(s,\chi) = \frac{1}{2\pi\mathrm{i}}\int_C \frac{L(z,\chi)}{(z-s)^2}\,\mathrm{d}z,$$

where C is some path convoluting s. We choose C to be the circle $\{z \mid |z-s| = (\log q)^{-1}\}$. This might cause us to leave the domain $\operatorname{Re}s > 1-(\log q)^{-1}$, however the bound of part b stays valid even if $\operatorname{Re}s > 1-2(\log q)^{-1}$. We get

$$L'(s,\chi) \ll \int_{|z-s|=(\log q)^{-1}} \frac{L(z,\chi)}{(z-s)^2}\,\mathrm{d}z \ll (\log q)^2.$$

Here we used $L(z,\chi) \ll \log q$ and $(s-z)^{-2} \ll (\log q)^2$, so the part in the integral is bounded by $O((\log q)^3)$. As we integrate over a path with length $O((\log q)^{-1})$, we obtain a bound with $O((\log q)^2)$, and we win.

Problem 3

Before solving this, we should maybe try to figure out why we would expect this result. Given some number n, we are supposed to evaluate the counting function

$$R(n) = \#\{p \le n \mid n-p \text{ is square free}\}.$$

Naively, one might be think that

$$R(n) \approx \zeta(2)^{-1}\pi(n) = \prod_p (1-p^{-2})\pi(n),$$

as the propability of a random number to be square-free is (in a suitable sense) given by $\zeta(2)^{-1}$, and we inspect numbers (which seem random) in a set of cardinality $\pi(n)$. This heuristic is not too far off, but it is wrong! The main term of the asymptotic is clearly different.
To see what goes wrong, let q be any prime number. First assume that $q \nmid n$. What is the probability that q^2 divides $n-p$ for some prime $p \neq q$? Neither n nor p are divisible by q, so the residue classes of these numbers mod q^2 are invertible, and there are $\varphi(q^2)$ such residue classes. So the probability is given by $\varphi(q^2)^{-1}$. Now assume $q \mid n$. One quickly checks that q^2 cannot divide $n-p$ (unless $p = q$, but this case does not contribute much). Now we can explain the asymptotic: There are $\approx \mathrm{Li}(n)$ primes $\leq n$, and the probability for $n-p$ not being divisible by some prime q is given by $(1-\varphi(q^2)^{-1})$ if $q \nmid n$ and by 1 if $q \mid n$. As $n-p$ is square-free iff no square of a prime divides it, we should expect

$$R(n) \approx \prod_{q\nmid n}(1-\varphi(q^2)^{-1})\mathrm{Li}(n) = \prod_{q\nmid n}\left(1-\frac{1}{q(q-1)}\right)^{-1}\mathrm{Li}(n),$$

and this is what we have to prove.

Proof. Clearly, we have $R(n) = \sum_{p\leq n}\mu^2(n-p)$. A standard trick to deal with μ^2 is writing it as $\mu(k) = \sum_{d^2|k}\mu(d)$. Applying this gives

$$R(n) = \sum_{p\leq n}\mu^2(n-p) = \sum_{p\leq n}\sum_{d^2|n-p}\mu(d) = \sum_{d\leq\sqrt{n}}\mu(d)\sum_{p\leq n,\ p\equiv n \bmod d^2} 1.$$

This is now basically an issue of counting primes in an arithmetic progression! Hence it really smells like Siegel-Walfisz, but this is not applicable right away. One issue is that we can only apply Siegel-Walfisz if $(d,n) = 1$. But restricting to those d does not really affect our main term, as whenever $(d,n) > 1$ there is at most one prime number in that arithmetic progression, and the contribution of those is bounded by $\omega(n) \ll n^{\varepsilon}$. Furthermore, and more seriously, Siegel-Walfisz is only applicable if d is small compared to n, more precisely, only if $d < (\log n)^A$. But again, we can elementarily bound the terms with $d > (\log n)^A$. Given some d, the amount of numbers $< n$ congruent to n mod d^2 can be bounded by $\ll \frac{n}{d^2}$. We obtain

$$R(n) = \sum_{d\leq(\log n)^A,\ (d,n)=1}\psi(n;n,d^2) + O\left(\sum_{(\log n)^A<d<\sqrt{n}}\frac{n}{d^2}\right) + O(\sqrt{n}),$$

and the O-terms can be bound by $\ll \frac{n}{(\log n)^A}$. Also, we can now apply Siegel-Walfisz! We find

$$R(n) = \sum_{d\leq(\log n)^A,\ (d,n)=1}\frac{1}{\varphi(d^2)}\mathrm{Li}(n) + O\left(\frac{n}{(\log n)^A}\right).$$

The sum can be completed, as $\varphi(d^2) \gg \frac{d^2}{\log\log d} \gg d^{2-\varepsilon}$, so that

$$\sum_{d>(\log n)^A}\frac{1}{\varphi(d^2)} \ll \frac{1}{(\log n)^{A(1-\varepsilon)}}.$$

This allows us to conclude (for any A, not the choice we made before)

$$R(n) = \sum_{d\in\mathbb{N},\ (d,n)=1}\frac{1}{\varphi(d^2)}\mathrm{Li}(n) + O_A\left(\frac{n}{(\log n)^A}\right) = \prod_{p\nmid n}\left(1-\frac{1}{\varphi(p^2)}\right)\mathrm{Li}(n) + O_A\left(\frac{n}{(\log n)^A}\right).$$

Problem 4

We follow the hint. Let $n \equiv 3 \mod 4$, write it as $n = 4k + 3$. Now

$$\frac{4}{n} - \frac{1}{k+1} = \frac{4}{n} - \frac{4}{n+1} = \frac{4}{n(n+1)} = \frac{4}{(4k+3)(4k+4)} = \frac{1}{(4k+3)(k+1)}.$$

This shows that there is a solution for every $n \equiv 3 \mod 4$. One also quickly verifies that if $\frac{4}{n} = \frac{1}{a} + \frac{1}{b}$, then $\frac{4}{mn} = \frac{1}{ma} + \frac{1}{mb}$. Also, there is a solution whenever n is even. Hence we really only have to show that almost all numbers have a prime divisor $\equiv 3 \mod 4$.

Now we can use (5.15). The numbers having only prime factors congruent 1 mod 4 is a subset of the numbers that can be written as a sum of two squares, and by (5.15), the number of sums of two squares up to x is bound by $O(\frac{x}{\sqrt{\log x}}) = o(x)$.

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Analytic Number Theory
Problem Set 11

Problem 1. Let

$$g(q) := \mu^2(q) \prod_{2<p|q} \frac{2}{p-2}.$$

Show (by elementary means or by Perron's formula or otherwise) that

$$\sum_{q \leqslant Q} g(q) \gg (\log Q)^2.$$

We want to develop another approach to Theorem 6.3. Recall the setup: let

$$S(\alpha) = \sum_{M<n \leqslant M+N} a_n e(\alpha n)$$

be a trigonometric polynomial. Let $\alpha_1, \ldots, \alpha_R \in \mathbb{R}$ be pairwise distinct in $\mathbb{R}/\mathbb{Z}$, and let $\delta := \min_{i \neq j} \|\alpha_i - \alpha_j\| > 0$. Our goal is to show that

$$\sum_{r \leqslant R} |S(\alpha_r)|^2 \ll (N + \delta^{-1}) \sum_n |a_n|^2. \tag{1}$$

Problem 2. a) Let c_{nr} be an $(N \times R)$ - matrix with complex entries, and let $D > 0$. Show that the following two statements are equivalent:

(A) $\sum_{r \leqslant R} \Big| \sum_{n \leqslant N} c_{nr} x_n \Big|^2 \leqslant D \sum_{n \leqslant N} |x_n|^2$ for all sequences x_n;

(B) $\sum_{n \leqslant N} \Big| \sum_{r \leqslant R} c_{nr} y_r \Big|^2 \leqslant D \sum_{r \leqslant R} |y_r|^2$ for all sequences y_r.

b) Show that (1) follows from

$$\sum_n f(n/N) \Big| \sum_{r \leqslant R} b_r e(n\alpha_r) \Big|^2 \ll (N + \delta^{-1}) \sum_r |b_r|^2 \tag{2}$$

where f is a fixed non-negative smooth function with $f(x) = 1$ on $[-1, 1]$ and $f(x) = 0$ for $|x| \geqslant 2$.

Hint for a: You can apply Cauchy-Schwarz for a direct proof, or you view (A) and (B) as bounds for an operator and its adjoint [one line].

Problem 3. Prove (2) by the Poisson summation formula.

Problem 4. Extend (6.4) to arithmetic progressions: for any sequence a_n and any two positive integers l, k show that

$$\sum_{\substack{q \leqslant Q \\ (k,q)=1}} \sideset{}{^*}\sum_{a \,(\mathrm{mod}\, q)} \Bigg| \sum_{\substack{M < n \leqslant M+N \\ n \equiv l \,(\mathrm{mod}\, k)}} a_n e\left(\frac{an}{q}\right) \Bigg|^2 \ll (Q^2 + Nk^{-1}) \sum_{n \equiv l \,(\mathrm{mod}\, k)} |a_n|^2.$$

Due: Tue, Jan 10

Zoom login for the lecture on Dec 23:

`https://uni-bonn.zoom.us/j/64362904429?pwd=S1FoL21rS2VYS2hodkc4NHZpa1Jndz09`
Meeting ID: 643 6290 4429
Passcode: 555095

Solutions to Sheet 11.

Problem 1

Let's first think about why this should be true. For $2 \neq p$ we have $g(p) = \frac{2}{p-2} \geq \frac{2}{p} = \frac{\tau(p)}{p}$. Hence for square-free numbers n we have $g(n) \geq \frac{\tau(n)}{n}$. It is easy to see that $\sum_{n \leq Q} \frac{\tau(n)}{n} \gg (\log Q)^2$ (approximate the LHS with $\left(\sum_{n \leq Q} \frac{\tau(n)}{n}\right)^2$). Hence we expect a similar lower bound (with a different constant) here. However, to make this precise we would have to show that the divisor function does not interact with the square-freeness condition too badly.

Proof using Perron's Formula. Let $G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ be the Dirichlet series attached to g. As g behaves similar to $\frac{\tau(n)}{n}$, we would hope to be able to relate g to $\zeta(s+1)^2$, which is the Dirichlet series attached to the coefficients $\frac{\tau(n)}{n}$. We write $G(s) = \zeta(s+1)^2 H(s)$, where we find in $\operatorname{Re} s > 0$

$$H(s) = \left(1 + \frac{1}{2^s}\right)\left(1 - \frac{1}{2^{s+1}}\right)^2 \prod_{p>2}\left(1 + \frac{2}{(p-2)p^s}\right)\left(1 - \frac{1}{p^{s+1}}\right)^2.$$

Factoring this out, we find that the euler factor at p is of size $1 + O(p^{-(s+2)})$, hence the euler product is absolutely (and locally uniformly) convergent whenever $\operatorname{Re} s > -1$, so H is a holomorphic function in that region and thereby does not interfere with the analysis when doing perron's formula. Also note that now G can be continued to $\operatorname{Re} s > -1$.

Now, we do what we always do. Let $T = x^\alpha$ (for some $\alpha \in (0,1)$) and $c = \frac{1}{\log x}$. We find by Perron's Formula

$$\sum_{n \leq x} g(n) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}T}^{c+\mathrm{i}T} G(s) x^s \frac{\mathrm{d}s}{s} + O\left(\frac{x^c}{T} \sum_{n \in \mathbb{N}} \frac{g(n)}{n^c} + \max_{n \sim x} g(n) \left(\frac{x \log x}{T}\right)\right).$$

We first inspect the O-term. As the series defining $G(s) = \zeta(s+1)^2 H(s)$ converges absolutely in $\operatorname{Re} s > 0$, we find that $g(n) \ll n^{-1+\varepsilon}$. As the pole of G at 0 has order 2, we have $\sum_{n \in \mathbb{N}} \frac{g(n)}{n^c} \ll (\log x)^2$. In particular, we find that the O-term is bounded by $O\left(\frac{x^\varepsilon}{T}\right)$.

We now want to shift the contour to the left, to $\operatorname{Re} s = -\frac{1}{8}$, say. We pick up a residue at $s = 0$. To compute the residue we develope everything into taylor series and find the residue to be of size $\frac{1}{2} H(0)(\log x)^2 + O(\log x)$. We obtain

$$\sum_{n \leq x} g(n) = \frac{1}{2} H(0)(\log x)^2 + \operatorname{Ver}(x,T) + \operatorname{Hor}(x,T) + O(\log x) + O\left(\frac{x^\varepsilon}{T}\right),$$

where $\operatorname{Ver}(x,T)$ denotes the integral along the vertical paths

$$\operatorname{Ver}(x,T) = \frac{1}{2\pi \mathrm{i}} \left(\int_{c-\mathrm{i}T}^{-1/8-\mathrm{i}T} + \int_{-1/8+\mathrm{i}T}^{c+\mathrm{i}T}\right) G(s) x^s \frac{\mathrm{d}s}{s}$$

and $\operatorname{Hor}(x,T)$ denotes the integral along the horizontal path

$$\operatorname{Hor}(x,T) = \frac{1}{2\pi \mathrm{i}} \int_{-1/8-\mathrm{i}T}^{-1/8+\mathrm{i}T} G(s) x^s \frac{\mathrm{d}s}{s}.$$

As $H(s)$ is absolutely bounded in $\operatorname{Re} s \geq -1/2$, so we can replace $G(s)$ by $\zeta(s+1)^2$ in all upcoming considerations. On the vertical lines, we have $x^s \ll x^{-1/8}$ and $\zeta(s+1)^2 \ll T^{1/4}$ (by the convexity bound), so that we find

$$\operatorname{Ver}(x,T) \ll TT^{1/4}x^{-1/8} = T^{5/4}x^{-1/8}.$$

(We could have also made use of the moment bounds, and improved this bound a lot by cutting the integral in dyadic pieces, but no need for that). For the horizontal integrals we use the convexity bound to find that

$$\frac{1}{s} \ll \frac{1}{T} \quad \text{and} \quad \zeta(s+1)^2 \ll T^{1/2} \quad \text{and} \quad x^s \ll 1,$$

revealing $\operatorname{Hor}(x,T) = O(1)$. If we choose $T = x^{\frac{1}{10}}$, we also find $\operatorname{Ver}(x,T) \ll 1$. Finally, note that $H(0) > 0$ (essentially by absolute convergence and the fact that no factor equals 0), so that

$$\sum_{n \leq x} g(n) = \frac{1}{2}H(0)(\log x)^2 + O(\log x) \gg (\log x)^2.$$

Elementary proof. Might be added later. See pages 179-181 in Brüdern's book.

Problem 2

a) If we consider C as a linear operator $\mathbb{C}^N \to \mathbb{C}^R$ and equip these spaces with the L^2-norm, the statement of the exercise is equivalent to the statement that the operator norm C and it's dual C^* coincide. This is a classical statement of functional analysis, and true in general for Hilbert spaces.

But just for the sake of completeness, here is a proof. It suffices to show that (A) implies (B), by symmetry. Assuming (A), we have

$$\text{LHS} = \sum_n \left| \sum_r c_{nr} y_r \right|^2 = \sum_{n,r,s} c_{nr}\overline{c_{ns}} y_r \overline{y_s} = \sum_r y_r \sum_n c_{nr} \sum_s \overline{c_{ns} y_s}.$$

Now we apply Cauchy-Schwartz to the sum over r, finding that

$$\text{LHS}^2 \leq \sum_r |y_r|^2 \sum_r \left| \sum_n c_{nr} \sum_s \overline{c_{ns} y_s} \right|^2 \leq \sum_r |y_r|^2 \, D \cdot \sum_n \left| \sum_s c_{ns} y_s \right|^2 = D \cdot \sum_r |y_r|^2 \cdot \text{LHS}.$$

b) We have to show that

$$\sum_r |S(\alpha_r)|^2 \ll (N + \delta^{-1}) \sum_n |a_n|^2$$

where $S(\alpha) = \sum_{M<n<M+N} a_n e(\alpha n)$ and the values α_r with pairwise distance at least δ. As in the proof from the lecture, we may shift by K without changing the absolute value of $S(\alpha)$, and may therefore assume $M \ll N$ (M might be negative). Of course we now want to apply part a), which leaves us with the task of showing that

$$\sum_{|n| \leq N} \left| \sum_r b_r e(n\alpha_r) \right|^2 \ll (N + \delta^{-1}) \sum_r |b_r|^2 .$$

Because opening the absolute values and estimating the inner sums turns out to be hard, we consider a smoothed version:

$$\sum_n f(n/N) \left| \sum_r b_r e(n\alpha_r) \right|^2 \ll (N + \delta^{-1}) \sum_r |b_r|^2 ,$$

where f is a non-negative function with $f|_{[0,1]} = 1$ and $f(x) = 0$ for $|x| > 2$. This clearly implies the bound above.

Problem 3

We open the square and interchange sums, obtaining

$$\sum_n f(n/N) \left| \sum_r b_r e(n\alpha_r) \right|^2 = \sum_{r,s} b_r \overline{b_s} \sum_n f(n/N) e(n(\alpha_r - \alpha_s)).$$

We use the elementary inequality $|ab| \le a^2 + b^2$ to obtain

$$\begin{aligned} \cdots \ll \sum_r \sum_s (|b_r|^2 + |b_s|^2) \left| \sum_n f(n/N) e(n(\alpha_r - \alpha_s)) \right| \\ = \frac{1}{2} \sum_r |b_r|^2 \sum_s \left| \sum_n f(n/N) e(n(\alpha_r - \alpha_s)) \right|, \end{aligned}$$

where in the latter inequality we used many symmetrys in this sum. We now try evaluating this. First, we consider the diagonal terms with $r = s$. Here we have $\alpha_r = \alpha_s$, and we easily find that this part of the sum is bounded by $\ll N \sum_r |b_r|^2$. For the remaining part, it suffices to show that

$$\sum_{s \neq r} \left| \sum_n f(n/N) e(n(\alpha_r - \alpha_s)) \right| \ll \frac{1}{\delta}.$$

The idea is that $\alpha_r - \alpha_s$ isn't too small, so we hope that there is cancellation in the sum. This is where Poisson's summation formula enters the stage. As f is Schwartz class function, its fourier transform is too and we find $\widehat{f}(y) \ll \frac{1}{1+y^2}$. Hence we obtain

$$\sum_n f(n/N) e(n(\alpha_r - \alpha_s)) = \sum_n \widehat{f}(N(\alpha_r - \alpha_s + n)) \ll N \sum_n \frac{1}{1 + N^2(\alpha_r - \alpha_s + n)^2}.$$

This is easily seen to be of size $\frac{N}{1+\|\alpha_r - \alpha_s\|^2 N^2}$. We are left to show that for fixed r,

$$\sum_{s \neq r} \frac{1}{1 + \|\alpha_r - \alpha_s\|^2 N^2} \ll \frac{1}{N\delta}.$$

It would not be good enough to just use that $\|\alpha_r - \alpha_s\| \gg \frac{1}{\delta}$. The only thing we can do to avoid this bound is to use that there are at most 2 values for s for which this is smaller than δ, at most four for which it is smaller than 2δ, etc. Hence we can bound the LHS as

$$\ll \sum_{n=1}^{\infty} \frac{1}{1 + n^2 \delta^2 N^2},$$

which leaves us with the task of showing that

$$\sum_{n=1}^{\infty} \frac{1}{1 + n^2 x^2} \ll \frac{1}{x}$$

whenever $x > 0$. This is one line:

$$\text{LHS} \ll \sum_{n=1}^{\infty} \min(1, \frac{1}{n^2 x^2}) \ll \sum_{n \le 1/x} 1 + \sum_{n \ge 1/x} \frac{1}{n^2 x^2} \ll \frac{1}{x} + \frac{1}{x^2} \sum_{n > 1/x} \frac{1}{n^2} \ll \frac{1}{x}.$$

Problem 4

The plan is to reduce this to (6.4). We can shift indices to assume that $l = 0$. Then we are summing over multiples of k in an interval of length N. This is the same as summing over integers in an interval of length N/k. The only thing that might be in our way is the exponential term, where we have the term $e(\frac{akd}{q})$, but we would like to have $e(\frac{ad}{q})$. But as we have $(k, q) = 1$, summing over a mod q is the same as summing over ka mod q. We arrive at something which really looks like (6.4), but with an additional coprimality condition. As all terms in the sum of (6.4) are positive, the inequality still stays valid, and we are done.

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Analytic Number Theory
Problem Set 12

Problem 1. Let $g(q) = \mu^2(q)\tau(q)/q$. Show that

$$\sum_{q \leqslant Q} g(q) \gg (\log Q)^2.$$

Problem 2. Use partial summation to conclude from (6.5) that

$$\sum_{\xi \leqslant q \leqslant Q} \frac{1}{\phi(q)} \sum_{\substack{\chi \,(\mathrm{mod}\, q) \\ \chi \text{ primitive}}} \Big| \sum_{n \leqslant x} w(n)\chi(n) \Big|^2 \ll \Big(\frac{x}{\xi} + Q\Big) \sum_{n \leqslant x} |w(n)|^2$$

for any $Q \geqslant \xi \geqslant 1$, $x \geqslant 1$ and any numbers $w(n) \in \mathbb{C}$.

Problem 3-4. The probability that $n \leqslant x$ is prime is about $1/\log x$. Hence we may conjecture that there are $\asymp \sqrt{x}/\log x$ primes $p \leqslant x$ of the form $n^2 + 1$. It is unknown if there are infinitely many such primes. Show with the large sieve that there are $\ll \sqrt{x}/\log x$ such primes. Proceed as follows:

a) Formulate the problem as a sieve problem.

b) Let $\rho(d)$ be the number of solutions of $\xi^2 + 1 \equiv 0 \,(\mathrm{mod}\, d)$. Compute $\rho(p)$ for primes $p \equiv 1 \pmod 4$, $p \equiv 3 \pmod 4$ and $p = 2$. Why is ρ multiplicative?

c) Apply (6.9), and show that $g(q) \geqslant \prod_{p \mid q} 2/p$ for q that are squarefree and consist only of primes $p \equiv 1 \pmod 4$. Conclude that $g(q)$ is majorized by the coefficients of $P(s) = \prod_{p \equiv 1 \,(\mathrm{mod}\, 4)} (1 + 2p^{-s-1})$.

d) Show that $P(s) = L(s+1, \chi_{-4})\zeta(s+1)H(s+1)$ where H is given by an Euler product that is absolutely convergent and uniformly bounded in $\Re s \geqslant 2/3$, say.

e) Use Perron or other means to show that $\sum_{q \leqslant x} g(q) \gg \log x$ (smoothing can simplify the analysis).

Due: Tue, Jan 17

Solutions to Sheet 12.

Problem 1

Again, we sketch a proof using Perron. We need to find out what the dirichlet function attached to g looks like. We have

$$G(s) = \sum_{n\in\mathbb{N}} g(n)n^{-s} = \prod_p \left(1 + \frac{2}{p^{s+1}}\right)$$

Remember that the Dirichlet series attached to $\frac{\tau(q)}{q}$ is given by

$$\zeta(s+1)^2 = \prod_p \left(1 + \frac{2}{p^{s+1}} + \frac{3}{p^{2(s+1)}} + \dots\right).$$

On each euler factor, the first two terms of $G(s)$ and $\zeta^2(s+1)$ coincide! So we might hope that there is a way to compare the two Dirichlet series. Indeed, writing

$$G(s) = \zeta(s+1)^2 H(s),$$

we find that $H(s)$ is given by an euler product with factor at p given by

$$\left(1 + \frac{2}{p^{s+1}}\right)\left(1 - \frac{1}{p^{s+1}}\right)^2 = 1 - \frac{3}{p^{2(s+1)}} + \frac{2}{p^{3(s+1)}} = 1 + O(p^{-2(s+1)}).$$

Now $H(s)$ is absolutely convergent and uniformly bounded in $\operatorname{Re} s \geq -\frac{1}{2} + \delta$, and we can copy the proof from exercise 1 on sheet 11.

Problem 2

Partial summation! Write

$$f(q) = \frac{1}{\varphi(q)} \left(\sum_{\chi(q)}^{*} \left|\sum_n \omega(n)\chi(n)\right|^2\right) \left(\sum_n |\omega(n)|^2\right)^{-1}.$$

Now (6.5) reads

$$\sum_{q\leq Q} qf(q) \ll N + Q^2.$$

By partial summation, we then find

$$\begin{aligned}
\sum_{R<q\leq Q} f(q) &\ll \frac{\sum_{R<q\leq Q} qf(q)}{Q} + \int_R^Q \frac{\sum_{R<q\leq t} qf(q)}{t^2}\, \mathrm{d}t \\
&\ll \frac{N+Q^2}{Q} + \int_R^Q \frac{N+t^2}{t^2}\, \mathrm{d}t \\
&\ll \frac{N}{Q} + Q + N\int_R^\infty t^{-2} dct \\
&\ll \frac{N}{R} + Q
\end{aligned}$$

and the claim follows.

Problem 3&4

a) There are (at least) two ways to set up the sifting problem. Either we sieve for those integers $n \le x^{1/2}$ such that n^2+1 is not divisible by prime numbers in some range, or we sieve for those integers $n \le x$ such that $n+1$ is a prime and n is a quadratic residue mod p for prime numbers in some different range. Let us think about the first idea, as this probably is what the exercise intents us to do. The other approach would probably be a good exercise though! We set

- $\mathcal{N} = \{x^{1/4} < n \le \sqrt{x}\}$ (this is the set of numbers we want to put into the sieve) (There is no particular reason to exclude numbers $\le x^{1/4}$, but there is also no reason to sieve for more, as we will soon see).
- $\mathcal{P} = \{2 \ne p \le x^{1/4}\}$ (this is the set of primes we want to sieve with) (this could have been chosen larger, but we will see why this is optimal (in some sense) soon).
- $\Omega_p = \{\text{Solutions to } a^2+1 \equiv 0 \bmod p\}$ (for each prime p, this is the set of residue classes mod p we want to throw out).

With this definition, we find that

$$\mathcal{N}^* = \{n \in (x^{1/4}, x^{1/2}] \mid \forall 2 \ne p \le x^{1/4} : p \nmid (n^2+1)\} \supset \{n \in (x^{1/4}, x^{1/2}] : n^2+1 \text{ prime}\}.$$

Hence, upper bounds for $\mathcal{N}^*$ deliver upper bounds for the number of primes of the form $p = n^2+1$ in the range $\sqrt{x} \le p \le x$. As there are $\ll \frac{\sqrt{x}}{\log x}$ primes up to $\sqrt{x}$, we further have

$$\#\{\text{primes of the form } p = n^2+1\} \ll \#\mathcal{N}^* + O\left(\frac{\sqrt{x}}{\log x}\right),$$

which shows that we need to show $\#\mathcal{N}^* \ll \frac{\sqrt{x}}{\log x}$ to finish the proof.

b) Note that $\rho(p) = \#\Omega_p$. We have $\rho(2) = 1$. Mod $p \ne 2$, there are 2 solutions to $\xi^2 \equiv -1$ mod p if $\left(\frac{-1}{p}\right) = 1$, i.e., if $p \equiv 1 \bmod 4$, and 0 otherwise. If $m = rs$ with $(r,s) = 1$ and we have $\rho(r)$ solutions $\xi_1^2 \equiv \cdots \equiv \xi_{\rho(r)}^2 \equiv -1 \bmod r$ and $\rho(s)$ solutions $\zeta_1^2 \equiv \cdots \equiv \zeta_{\rho(s)}^2 \equiv -1 \bmod s$, then by the chinese remainder theorem (and the fact that $a \equiv -1 \bmod m$ iff $a \equiv -1 \bmod r$ and $a \equiv -1 \bmod s$) we find that there are $\rho(r)\rho(s)$ solutions mod rs. This shows $\rho(rs) = \rho(r)\rho(s)$, as desired.

c) Using (6.9), we can bound the number of elements in $\mathcal{N}^*$. We put

$$g(q) := \mu(q)^2 \prod_{p \mid q} \frac{\omega(p)}{p - \omega(p)},$$

where $\omega(p) = \rho(p)$ if $p \in \mathcal{P}$ and 0 otherwise. In particular, g vanishes on even numbers. Now we have for any $Q > 1$

$$\#\mathcal{N}^* \ll (N + Q^2) \left(\sum_{q \le Q} g(q)\right)^{-1},$$

where $N = \#\mathcal{N} \ll \sqrt{x}$. The task is to find a lower bound for

$$\sum_{q \le Q} g(q),$$

where (in order not to disturb the main term) we will choose $Q \le x^{1/4}$ (this is also why it suffices to only consider $p \le x^{1/4}$: these are the only prime divisors that occur as divisors of numbers $\le Q$). As ρ (and hence ω too) is multiplicative, we find that whenever q only has prime divisors $\equiv 1$ mod 4 that are in $\mathcal{P}$,

$$g(q) = \prod_{p|q} \frac{2}{p-2} \ge \prod_{p|q} \frac{2}{p}.$$

As $g(q)$ is supported on numbers that only have prime divisors $p \in \mathcal{P}$ with $\equiv 1$ mod 4, we find that for such numbers we have $g(q) \ge l(q)$, where $l(q)$ is implicitely defined via

$$P(s) = \prod_{p\equiv 1 (\mathrm{mod}\ 4)} (1 + 2p^{-s-1}) = \sum_{n\in\mathbb{N}} l(n) n^{-s}.$$

If we assume to know that $\sum_{q\le Q} l(q) \gg \log Q$, we find for $Q \le x^{1/4}$ that

$$\sum_{q\le Q} g(q) \gg \sum_{q\le Q} l(q) \gg \log Q$$

and hence

$$\#\mathcal{N}^* \ll (\sqrt{x}) \left(\sum_{q\le x^{1/4}} g(q) \right)^{-1} \ll \frac{\sqrt{x}}{\log x},$$

which is what we wanted to show. (I don't think we need this, however).

d) It remains to show that indeed $\sum_{q\le Q} l(q) \gg \log Q$. Our tool of choice will be Perron's formula again, or some variant thereof with smooth weights (elementary proofs are certainly possible, but probably less constructive). Let's choose c, T and write down what (4.7) says:

$$\sum_{q\le Q} l(q) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}T}^{c+\mathrm{i}T} P(s) Q^s \frac{\mathrm{d}s}{s} + O\left(\frac{Q^c}{T} \sum_n \frac{|l(q)|}{n^c} + \max_{q\sim Q} |l(q)| \left(1 + \frac{Q \log Q}{T}\right)\right). \quad (1)$$

One of the main tasks is now to express $P(s)$ in a way that makes it possible to calculate its analytic behaviour. The hint tells us that perhaps

$$P(s) \approx L(\chi_{-4}, s+1)\zeta(s+1),$$

which is nice because we know how to deal with $\zeta(s+1)$ and $L(\chi_{-4}, s+1)$. Indeed, the Euler factor at $p \equiv 1 \pmod 4$ of $L(s+1, \chi_{-4})\zeta(s+1)$ is given by

$$(1 + p^{-(s+1)} + p^{-2(s+1)} + \cdots)^2 = 1 + 2p^{-s-1} + O(p^{-2(s+1)})$$

and at $p \equiv 3$ mod 4 we find

$$(1 + p^{-(s+1)} + p^{-2(s+1)} + \dots)(1 - p^{-(s+1)} + p^{-2(s+1)}) = 1 + O(p^{-2(s+1)}).$$

We can use the power series expansion

$$(1+x)^{-1} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

to deduce that the euler factors of $H(s+1) = P(s)(\zeta(s+1)L(\chi_{-4}, s+1)^{-1}$ all lie in $1 + O(p^{-2(s+1)})$, which gives that $H(s)$ is holomorphic and absolutely convergent (hence uniformly bounded) in $\mathrm{Re}\, s \ge 2/3$.

e) First, we will work through how one can find the bound using (1). Then we will discuss how one could have used a smooth weight to simplify the analysis.

Okay, so let's start with (1). As usual, we choose $c = 1/\log Q$ and $T = Q^\alpha$ for some $\alpha \in (0,1)$. We first inspect the O-term. As $P(s)$ is absolutely convergent in $\operatorname{Re} s > 0$, we find that $l(q) \ll q^{\varepsilon - 1}$. As $L(\chi_{-4}, 1) \neq 0$, $P(s)$ has at most a simple pole at 0, hence we find $\sum_n g(n) n^{-c} \ll Q^\varepsilon$, and the whole O-term is bounded by $O(Q^\varepsilon/T)$. By the product expansion and the analytic continuations of ζ and L, we can continue P to a meromorphic function in $\operatorname{Re} s > -1/3$, and we know that the only pole is at $s = 0$ with residue $H(1)L(\chi_{-4}, 1) \neq 0$. We find that

$$\operatorname{Res}_{s=0}\left(P(s)\frac{Q^s}{s}\right) = H(1)L(\chi_{-4}, 1)(\log Q) + C$$

for some constant C independent of Q. Now we have to shift the contour, and every contour a tad to the left of $\operatorname{Re} s = 0$ suffices. Hence we might choose $\operatorname{Re} s = -1/8$. The remaining integral along the path $\gamma_1 \cup \gamma_2 \cup \gamma_3$ where

$$\gamma_1 = [c - \mathrm{i}T, -1/8 - \mathrm{i}T], \quad \gamma_2 = [-1/8 - \mathrm{i}T, -1/8, \mathrm{i}T], \quad \gamma_3 = [-1/8 + \mathrm{i}T, c + \mathrm{i}T]$$

can be easily bounded using the convexity bound, which states that in this region

$$\zeta(s) \ll (1 + |s|)^{\frac{1-\sigma}{2} + \varepsilon} \quad \text{and} \quad L(s, \chi_{-4}) \ll (1 + |s|)^{\frac{1-\sigma}{2} + \varepsilon}.$$

In total, after choosing T (more precisely, α) appropiately small, no integal contributes more than $O(1)$. This shows the asymptotic

$$\sum_{q \le Q} l(q) = H(1)L(1, \chi_{-4})(\log Q) + O(1),$$

and we in particular find $\sum_{q \le Q} l(q) \gg \log Q$.

Using a smooth weight. We can make our life a lot easier if we choose some smooth weight ω with support in $[0,1]$ and $\omega|_{[0,1/2]} = 1$. With this choice, the derivative of ω is compactly supported. Note that by integration by parts and in $\operatorname{Re} s > 0$ we have

$$\widehat{\omega}(s) = \int_0^\infty \omega(x) x^{s-1}\,\mathrm{d}x = -\frac{1}{s}\int_0^\infty \omega'(x) x^s\,\mathrm{d}x = -\frac{1}{s}\mathcal{M}(\omega')(s). \tag{2}$$

Here, $\mathcal{M}(\omega')$ is holomorphic on $\mathbb{C}$ and rapidly decaying on vertical lines by (4.4). Therefore, (2) gives a meromorphic continuation of $\widehat{\omega}$ to $\mathbb{C}$ with a simple pole at 0, and we find that $\widehat{\omega}$ is also rapidly decaying on vertical lines.

We find

$$\sum_{q \le Q} l(q) \ge \sum_{q \in \mathbb{N}} l(q)\omega(q/Q) = \frac{1}{2\pi \mathrm{i}}\int_{(c)} P(s) Q^s \widehat{\omega}(s)\,\mathrm{d}s.$$

This integral is converging absolutely. Now shifting the integral to the left is easy as $\widehat{\omega}$ is eating through everything (note that ζ and L dont grow too fast by the convexity bound) and the horizontal integrals vanish in $\lim_{T\to\infty}$. We find

$$\frac{1}{2\pi \mathrm{i}}\int_{(c)} P(s) Q^s \widehat{\omega}(s)\,\mathrm{d}s = \frac{1}{2\pi \mathrm{i}}\int_{(-1/8)} P(s) Q^s \widehat{\omega}(s)\,\mathrm{d}s + \operatorname{Res}_{s=0}\left(P(s) Q^s \widehat{\omega}(s)\right).$$

As before, the residue is of size $\gg \log Q$, and the remaining integral is absolutely convergent, thereby of size $O(Q^{-1/8})$.

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Analytic Number Theory
Problem Set 13

Problem 1. Prove (7.6).

Problem 2. Prove (8.6).

Problem 3.-4. Let χ be a primitive Dirichlet character modulo $q > 2$. The proof of (5.13) showed cancellation between Λ and χ, namely

$$\sum_{n \leqslant x} \Lambda(n)\chi(n) \ll x \exp(-c\sqrt{\log x}) \tag{1}$$

for $q \leqslant (\log x)^A$ as a consequence of the zerofree region of $\frac{L'}{L}(s,\chi)$. One would expect that this also implies cancellation between μ and χ as a consequence of the same zerofree region of $1/L(s,\chi)$. The purpose of this exercise is to make this precise without using complex analysis.

a) Conclude from (1) that

$$\sum_{n \leqslant x} \Lambda(n)\chi(n) \ll_{\delta,B} q^{\delta} x/(\log x)^B$$

for all $\delta, B > 0$ and all $q > 2$. Conclude that

$$\sum_{p \leqslant x} \chi(p) \ll_{\delta,B} q^{\delta} x/(\log x)^B$$

for all $\delta, B > 0$ and all $q > 2$.

b) Conclude that

$$\sum_{n \leqslant x} \chi(n)\mu(n) \ll_{\delta,B} q^{\delta} x/(\log x)^B$$

for all $\delta, B > 0$ and all $q > 2$.

c) Conclude that $D_{\mu}(x,q,a)$ is $(\log x)^A$-distributed in the sense of (8.3) for every $A > 0$.

Hint for (b): Write $n = mp$ where p is the largest prime factor of n. Justify why

$$\sum_{n \leqslant x} \chi(n)\mu(n) = 1 - \sum_{m \leqslant x} \mu(m)\chi(m) \sum_{p_m < p \leqslant x/m} \chi(p)$$

where p_m denotes the largest prime factor of m. Estimate the inner sum with part a). To this end, show that the summation conditions imply $m \leqslant x^{\omega(m)/(\omega(m)+1)}$, so that

$\log(x/m) \geqslant (\omega(m) + 1)/\log x$.

Due: Tue, Jan 24

Solutions to Sheet 13.

Problem 1

I somehow couldn't make Selberg's sieve (7.5) work, so we will just use the large sieve (6.9) again. We set $F(x) = \prod_{j=1}^{k}(q_j x + r_j)$.

First we need to set up the large sieve. We make the following choices:

- $\mathcal{N} = \{n \leq x\}$
- $\mathcal{P} = \{p \leq \sqrt{x}\}$
- $\Omega_p = \{a \in \mathbb{Z}/p\mathbb{Z} \mid F(a) \equiv 0 \pmod{p}\}$.

With this setup, we sift for those $n \leq x$ such that all the numbers $q_j n + r_j$ have no prime divisors $\leq \sqrt{x}$, i.e.,

$$\#\mathcal{N}^* = \{n \leq x : p \mid q_j n + r_j \implies p > \sqrt{x}\}.$$

As $q_j n + r_j$ might be larger than x, this does not guarantee that indeed all remaining numbers are primes, but we still get an upper bound, so we are good. There are some subteleties with this setup. First, note that it might happen that $\omega(p) = \#\Omega_p = p$. In this case however we find that $\mathcal{N}^* = \emptyset$, and any upper bound holds. Our long-term goal is the following. As $\deg(F) = k$, we expect $\omega(p) = k$ for *most* primes p, and hence we should have something like

$$g(m) = \mu^2(m) \prod_{p \mid m} \frac{\omega(p)}{p - \omega(p)} \approx \mu^2(m) \prod_{p \mid m} \frac{k}{p} \approx \mu^2(m) \frac{\tau_k(m)}{m}.$$

From here we want to proceed as on the previous sheets (using Perron's formula) to show that

$$\sum_{m \leq x} g(m) \gg (\log x)^k.$$

And indeed, we can show that *most* means *all but finitely many.* First, we throw out all p that divide one of the q_j. Then for all remaining p, there is for every j a unique residue a_j such that $p \mid q_j a_j + r_j$. If F had a multiple zero mod p, we'd have that $a_j = a_i$ for some $j \neq a_i$, which immediately implies that

$$p \,\Big|\, \det \begin{pmatrix} q_j & q_j \\ r_i & r_j \end{pmatrix}.$$

But all those determinants are non-vanishing, so that there is only a finite number of such p. Hence we set $\mathcal{P}' = \mathcal{P} \setminus \{\text{finite set of primes}\}$, and we can apply the methods the last two sheets to find lower bounds for $\sum_{n \leq x} g(n)$. Everything goes through, as removing a finite set of primes only changes the Pirichlet function we consider in the Perron approach by some finite euler product (which thereby is holomorphic). (Although, if we write $G(s) = \zeta(s+1)^k H(s)$ with $H(s)$ absolutely convergent a bit to the left of $\operatorname{Re} s = 0$, we would still have to somehow show that $H(0) \neq 0$. But I guess this can be done elementarily).

Problem 2

I am a bit annoyed that this is in the exercises, this is purely elementary. But here we go.

Given a function f which is continuous and monotonic, we have to show that

$$|D_f(x;q,a)| = \left| \sum_{n\leq x \ n\equiv a (\text{mod })q} f(n) - \frac{1}{\varphi(q)} \sum_{n\leq x, \ (n,q)=1} f(n) \right| \leq 2(|f(1)| + |f(x)|). \tag{1}$$

By symmetry, we may assume that f is positive and monotonely increasing. We split up $D_f(x;q,a)$, writing $K = \lfloor \frac{x-a}{q} \rfloor - 1$:

$$D_f(x;q,a) = -\frac{1}{\varphi(q)} \sum_{n<\min(a,x), \ (n,q)=1} f(n) + \sum_{k=0}^{K-1} \left(f(kq+a) - \frac{1}{\varphi(q)} \sum_{kq+a\leq n<(k+1)q+a} f(n) \right)$$
$$+ f((K+1)q+a)\delta_{x\geq a} - \frac{1}{\varphi(q)} \sum_{\lfloor \frac{x-a}{q} \rfloor q+a\leq n\leq x \ (n,q)=1} f(n).$$

The first and the last sum have combined size $\geq -(f(1)+f(x))$, together with the summand $f((K+1)q+a)\delta_{x\geq a}$ we find that everything except the big sum in the middle is of absolute size $\leq (f(1)+f(x))$. This is easily seen by monotonicity of f and the fact that these sums run over sets of combined cardinality at most $\varphi(q)$. The sum in the middle telescopes. Indeed, by monotonicity we have

$$f(kq+a) \leq \frac{1}{\varphi(q)} \sum_{kq+a\leq n<(k+1)q+a} f(n) \leq f((k+1)q+a),$$

and these inequalities can be combined into the bound

$$0 \geq \sum_{k=0}^{\lfloor \frac{x-a}{q} \rfloor} \left(f(kq+a) - \frac{1}{\varphi(q)} \sum_{kq+a\leq n<(k+1)q+a} f(n) \right) \geq -\frac{1}{\varphi(q)} \sum_{Kq+a\leq n<(K+1)q+a} f(n).$$

The RHS is $\geq -f(x)$. Combining everything, we showed

$$|D_f(x;q,a)| \leq f(1) + 2f(x) \leq 2(f(1)+f(x)).$$

Problem 3&4

By (5.13), we have for any $A>0$ any $q<(\log x)^A$ and any primitive χ mod q that

$$\sum_{n\leq x} \Lambda(n)\chi(n) \ll_A x\exp(-c\sqrt{\log x}). \tag{2}$$

For any $C>0$, this can be relaxed to

$$\sum_{n\leq x} \Lambda(n)\chi(n) \ll_A x\exp(-c\sqrt{\log x}) \ll x/(\log x)^C. \tag{3}$$

a) We first show that

$$\sum_{n\leq x} \Lambda(n)\chi(n) \ll_{\delta,B} q^\delta x/(\log x)^B \tag{4}$$

for all $\delta, B > 0$ and $q > 2$. First, note that this claim is a direct consequence of the prime number theorem once $q > (\log x)^{B/\delta}$. Hence it suffices to check this for $q < (\log x)^{B/\delta}$. But in this region we can apply (3) with $A = B/\delta$ and $C = B$. We obtain the inequalities

$$\sum_{n\leq x} \Lambda(n)\chi(n) = \begin{cases} O_{B/\delta}(x/(\log x)^B) & q < (\log x)^{B/\delta} \\ O(x) & q \geq (\log x)^{B/\delta}. \end{cases}$$

It is easily seen that these inequalities combine into (4).

Now we have to show that

$$\sum_{p\leq x} \chi(p) \ll_{\delta,B} q^\delta x/(\log x)^B. \tag{5}$$

This is akin to deriving the prime number theorem for π from the prime number theorem for ψ. Namely, it is another exercise in partial summation. This will just be a sketch of the arguments, look at sheet 8 for more details. First note that the higher prime powers do not contribute much, we have

$$\sum_{n\leq x} \Lambda(n)\chi(n) = \sum_{p^k\leq x} (\log p)\chi(p^k) = \sum_{p\leq x} (\log p)\chi(p) + O(x^{1/2+\varepsilon}). \tag{6}$$

Now, partial summation. We find

$$\sum_{p\leq x} \chi(p) = (\log x)^{-1} \sum_{p\leq x} (\log p)\chi(p) + \int_1^x \frac{\sum_{p\leq t}(\log p)\chi(p)}{t(\log t)^2}\, \mathrm{d}t.$$

Repeated use of (6) and (4) yields the claim: The integral can be bounded by

$$\ll q^\delta \int_1^x \frac{1}{(\log t)^{2+B}}\, \mathrm{d}t \leq q^\delta \left(x^{1/2} + \int_{x^{1/2}}^x \frac{2^B}{(\log x)^B}\, \mathrm{d}t \right) \ll_B q^\delta x(\log x)^{-B},$$

the remaining term is easily seen to be of that size too.

Remark. This bound is even true for all $\chi \neq \chi_0 \bmod q$. Indeed, suppose non-principal $\chi \bmod q$ is implied by $\chi_1 \bmod q_1$, with χ_1 primitive. We can compare the sums over χ and χ_1, as

$$\sum_{n\leq x} \chi(n)\Lambda(n) = \sum_{n\leq x,\ (n,q)=1} \chi(n)\Lambda(n) + O(\omega(q)(\log x)^2),$$

and in the latter sum we can replace χ with χ_1. We also have

$$\sum_{n\leq x} \chi_1(n)\Lambda(n) \ll_{\delta,B} q_1^\delta x(\log x)^{-B} \ll q^\delta x(\log x)^{-B},$$

and the claim follows after combining the previous two equations with (4).

b) Now we are supposed to show that

$$\sum_{n\leq x} \chi(n)\mu(n) \ll_{\delta,B} q^\delta x/(\log x)^B. \tag{7}$$

As the hint commands, we try to make use of the bijection

$$\{\text{square-free numbers } n \leq x\} \leftrightarrow \{(m,p) \mid mp \leq x \wedge p_m < p \wedge m\ \square\text{-free}\}.$$

Here (and from now on), p_m denotes the largest prime divisor of m. Let's just insert this and see what we get.

$$\sum_{n\leq x} \mu(n)\chi(n) = \sum_{n\leq x\ \square\nmid n} \mu(n)\chi(n) = 1 - \sum_{m\leq x} \chi(m)\mu(m) \sum_{p_m\leq p\leq x/m} \chi(p).$$

Applying (5) to the inner sum and using that that $\mu(m)\chi(m) \ll 1$ (and writing $\sum_{p_m \le p \le x}$ as $\sum_{p\le x} - \sum_{p \le p_m}$) yields

$$\text{RHS} \ll_{\delta,B} 1 + \sum_{mp_m \le x,\ \square \nmid m} q^\delta \frac{x}{m} (\log \tfrac{x}{m})^{-B}.$$

Our main task is now to bound $(\log \frac{x}{m})$. As $mp_m \le x$ and $m \le p_m^{\omega(m)}$ (note that m is square-free), we find that $m^{\frac{\omega(m)+1}{\omega(m)}} \le x$, which implies that $x^{\frac{1}{\omega(m)+1}} \le \frac{x}{m}$. As $B > 0$, this implies that

$$\text{RHS} \ll 1 + \sum_{\square \nmid m \le x} q^\delta \frac{x}{m} (\log x)^{-B} (1+\omega(m))^B$$

We use that $(1+\omega(m))^B \le 2^B \omega(m)^B \ll_B \omega(m)^B$. Now we only need to find a bound for the sum

$$\sum_{\square \nmid m \le x} \frac{\omega(m)^B}{m}.$$

I struggeled very hard with bounding this, at some point Bart thold me how it's done: We show that

$$\sum_{\square \nmid m \le x} \frac{\omega(m)^B}{m} \ll (\log x)(\log\log x)^B.$$

For $B = 0$, this is obvious. We now do induction on B, and basically just reorder the sum.

$$\sum_{\square \nmid m \le x} \frac{\omega(m)^B}{m} \le \sum_{m \le x} \frac{\omega(m)^{B-1}}{m} \sum_{p|m} 1 = \sum_{p \le x} \frac{1}{p} \sum_{m \le x/p} \frac{\omega(m)^{B-1}}{m}$$
$$\ll \sum_{p\le x} \frac{1}{p} (\log x)(\log\log x)^{B-1} \ll (\log x)(\log\log x)^B.$$

In the last step we used Merten's theorem for the sum of reciprocals of the primes. This finishes the proof of (7).

c) The first challenge is to even find out what we are supposed to show. We will show that

$$D_\mu(x;q,a) := \sum_{n\le x\ n \equiv a \bmod q} \mu(n) - \frac{1}{\varphi(q)} \sum_{n \le x} \mu(n) \ll x(\log x)^{-A}, \tag{8}$$

which is similar to (8.3) as $\|\mu\|_2 = \sqrt{\sum_{n\le x} \mu(n)^2} \approx x^{1/2}$. The function $D_\mu(x;q,a)$ measures how far μ fails to be equidestributed in the residue class a mod q up to x. Trickery with orthogonality relations quickliy reveals

$$D_\mu(x;q,a) = \frac{1}{\varphi(q)} \sum_{\chi \ne \chi_0} \overline{\chi}(a) \sum_{n\le x} \mu(n)\chi(n), \tag{9}$$

this form was already hinted at in (8.3). And this is really nice, as now we can make use of the previous parts of the exercise. If we just insert (7) in (9), we directly obtain

$$D_\mu(x;q,a) \ll_{B,\delta} q^\delta x (\log x)^{-B}.$$

This is not strong enough, as the bound in (8) should be uniform in q, and the RHS explodes if q is large. However, we aren't far from solving this exercise. Let $A > 0$ be given. In the range $q < (\log x)^{2A}$, the previous inequality gives the desired bound (choose $\delta = 1$ and $B = 3A$). If q

is larger, there are only a few numbers we sum over. Indeed, if $q > (\log x)^{2A}$, we can trivially bound from the definition (8):

$$\sum_{n \le x \ n \equiv a \bmod q} \mu(n) - \frac{1}{\varphi(q)} \sum_{n \le x} \mu(n) \ll 1 + \frac{x}{q} + \frac{\log \log q}{q} x \ll x(\log x)^{-A}.$$

(Remember that $\varphi(q) \gg \frac{q}{\log \log q} \gg q^{1-\varepsilon}$ for every $\varepsilon > 0$). These two cases combine into the desired bound. GGWP.

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